Slides week 21
Tensor Analysis

Lecture 2

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2024/25

Chapter 3: Local Coordinate Transform

Today: Chapter 3–Local Coordinate Transform

1. Preliminaries,
2. Dual bases,
3. Covariant and contravariant components of a vector,
4. The transformation rule,
5. The relationship between covartiant and contravariant components, and
6. Arc length and the metric tensor.

Reminder

Definition of Coordinate Systems

Coordinate systems are defined by a set of **basis vectors**.

Definitions

- A coordinate system is **orthogonal** if its basis vectors intersect at 90° angles, i.e.,

$$\mathbf{v}_i \cdot \mathbf{v}_j = 0, \text{ for } i \neq j.$$

- A coordinate system is **orthonormal** if it is orthogonal and its basis vectors have magnitude 1, i.e.,
  - $\mathbf{v}_i \cdot \mathbf{v}_j = 0$, for $i \neq j$, and
  - $|\mathbf{v}_i| = \sqrt{\mathbf{v}_i \cdot \mathbf{v}_i} = 1$.

Cartesian and generalised coordinate systems

Cartesian coordinate system

The **Cartesian coordinate** system is an example of **orthonormal coordinate system.**

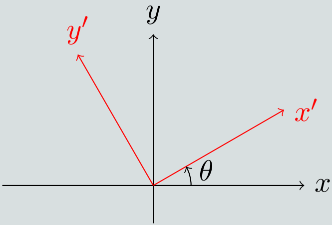
$$\mathbf{i} = (1, 0, 0), \quad \mathbf{j} = (0, 1, 0), \quad \mathbf{k} = (0, 0, 1)$$

Definition

Generalised coordinate systems do not necessarily have orthogonal coordinate curves.

Cartesian coordinate system in 2D

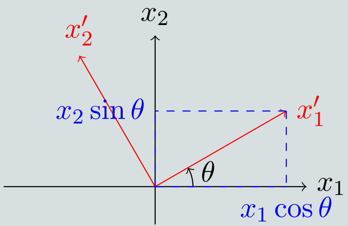
- Define a $2D$ coordinate system by the plane (x_1, x_2).
- Then, rotate (x_1, x_2) by some angle θ to obtain a new coordinate system (x'_1, x'_2).
- How do we describe vectors in this new coordinate system?



Cartesian coordinate system in 2D

Any point P in (x_1, x_2) is related to a point in (x'_1, x'_2) via

$$x'_1 = x_1 \cos\theta + x_2 \sin\theta,$$
$$x'_2 = x_2 \cos\theta - x_1 \sin\theta,$$



In matrix form

$$\begin{pmatrix} x'_1 \\ x'_2 \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

Definition.

Define the **rotation matrix** by

$$L = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix}.$$

Cartesian coordinate system in 2D

The new coordinate system (x_1', x_2') obtained by rotating (x_1, x_2) is

$$x_1' = L_{11}x_1 + L_{12}x_2 = L_{1j}x_j,$$
$$x_2' = L_{21}x_1 + L_{22}x_2 = L_{2j}x_j,$$

Or more compactly in suffix notation:

$$x_i' = L_{ij}x_j.$$

Properties of the Rotation Matrix

Some important properties of the rotation matrix

$$L = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

are

- $L^{-1} = L^T$.
  - This means that $LL^T = I$ and $L^T L = I$, thus in suffix notation we have

$$L_{ij}L^T_{jk} = \delta_{ik} \quad \text{that is} \quad L_{ij}L_{kj} = \delta_{ik}$$
$$L^T_{ij}L_{jk} = \delta_{ik} \quad \text{that is} \quad L_{ji}L_{jk} = \delta_{ik}.$$

- Its determinant is

$$|L| = \cos^2\theta + \sin^2\theta = 1.$$

Transformation rules

If a vector $\mathbf{v}$ has

- components (v_1, v_2) in the coordinate system (x_1, x_2),
- its components (v'_1, v'_2) in the coordinate system (x'_1, x'_2) (obtained by rotating (x_1, x_2)) are given by the formula

$$x'_i = L_{ij} x_j.$$

- The inverse transformation is

$$x_i = L_{ji} x'_j.$$

Chapter 3: Preliminaries - Dimension 3

What about dimension 3?

- Let $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ be basis vectors.
- The position vector in this coordinate system is:

$$\mathbf{r} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3 = \mathbf{e}_j x_j.$$

- Suppose we rotate the basis vectors, obtaining new basis vectors $\mathbf{e}'_1$, $\mathbf{e}'_2$, and $\mathbf{e}'_3$.
- Then the position vector in this new coordinate system is

$$\mathbf{r}' = x'_1\mathbf{e}'_1 + x'_2\mathbf{e}'_2 + x'_3\mathbf{e}'_3 = \mathbf{e}_j x_j.$$

- Can we find a formula for x'_i in terms of the x_k? And vice versa?

Transformation in 3D

- Let $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ be basis vectors.
- Let $\mathbf{e}'_1$, $\mathbf{e}'_2$, and $\mathbf{e}'_3$ be new basis vectors, obtained from rotating the basis $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ by θ degrees.
- The position vectors in these coordinate system are:

$$\mathbf{r} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3 = \mathbf{e}_j x_j,$$
$$\mathbf{r}' = x'_1\mathbf{e}'_1 + x'_2\mathbf{e}'_2 + x'_3\mathbf{e}'_3 = \mathbf{e}'_j x'_j.$$

- We have

$$x'_i = \mathbf{e}'_i \cdot \mathbf{r} = \mathbf{e}'_i \cdot (\mathbf{e}_j x_j) = (\mathbf{e}'_i \cdot \mathbf{e}_j) x_j.$$

Transformation in 3D

In the previous slide, we have shown

$$x'_i = \mathbf{e}'_i \cdot \mathbf{r} = (\mathbf{e}'_i \cdot \mathbf{e}_j)x_j.$$

Thus, we conclude that the transformation matrix is $L = (L_{ij})$with entries given by

$$L_{ij} = \mathbf{e}'_i \cdot \mathbf{e}_j.$$

Remark

The matrix L is called the **transformation matrix** because it allows you to switch between the coordinate systems (x_1, x_2) and (x'_1, x'_2) using the formula

$$x'_i = L_{ij}x_j.$$

Transformation matrix

When we say "switch between the coordinate systems (x_1, x_2, x_3) and (x_1', x_2', x_3')" we mean:

- Let $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ be a basis for the coordinate system (x_1, x_2, x_3).
- Let $\mathbf{e}_1'$, $\mathbf{e}_2'$, $\mathbf{e}_3'$ be a basis for the coordinate system (x_1', x_2', x_3').
- By the definition of basis, we can expand **any** vector $\mathbf{v}$ in terms of these bases. That is,

$$\mathbf{v} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + v_3\mathbf{e}_3, \text{ and}$$
$$\mathbf{v} = v_1'\mathbf{e}_1' + v_2'\mathbf{e}_2' + v_3'\mathbf{e}_3',$$

for some constants v_i and v_i'.

Transformation matrix– continuation

- In particular, we can expand the vectors $\mathbf{e}'_1$, $\mathbf{e}'_2$, $\mathbf{e}'_3$ in terms of $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$:

$$\mathbf{e}'_i = \lambda_{i1}\mathbf{e}_1 + \lambda_{i2}\mathbf{e}_2 + \lambda_{i3}\mathbf{e}_3,$$

  for some constants λ_{i1}, λ_{i2}, λ_{i3}.

- The fact that the transformation between (x_1, x_2, x_3) and (x'_1, x'_2, x'_3) is given by

$$x'_i = L_{ij}x_j$$

  is telling us precisely that the coefficients of the expansion of $\mathbf{e}'_i$ are the entries of L:

$$\mathbf{e}'_i = L_{i1}\mathbf{e}_1 + L_{i2}\mathbf{e}_2 + L_{i3}\mathbf{e}_3, \quad i = 1, 2, 3.$$

Properties of the transformation matrix

- The number L_{ij} is simply the cosine of the angle between $\mathbf{e}'_i$ and $\mathbf{e}_j$.
- L is **orthogonal** i.e. $L^T L = I = LL^T$. In particular,

$$L_{ij}L_{kj} = \delta_{ik} = L_{ji}L_{jk}.$$

- From $x'_i = L_{ij}x_j$ and $x_i = L_{ji}x'_j$, we can derive

$$\frac{\partial x'_i}{\partial x_j} = L_{ij} \qquad \text{and} \qquad \frac{\partial x_i}{\partial x'_j} = L_{ji}.$$

We will show this in the next slide.

Properties of the transformation matrix

We want to show the formulas

$$\frac{\partial x'_i}{\partial x_j} = L_{ij} \qquad \text{and} \qquad \frac{\partial x_i}{\partial x'_j} = L_{ji}.$$

In fact,

$$x'_i = L_{ik}x_k = \sum_{k=1}^{3} L_{ik}x_k$$

implies

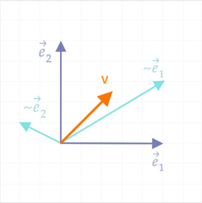
$$\frac{\partial x'_i}{\partial x_j} = \sum_{k=1}^{3} \frac{\partial (L_{ik}x_k)}{\partial x_j} = \sum_{k=1}^{3} \left(\frac{\partial L_{ik}}{\partial x_j} x_k + L_{ik} \frac{\partial x_k}{\partial x_j} \right)$$
$$= \sum_{k=1}^{3} L_{ik}\delta_{kj} = L_{ij}.$$

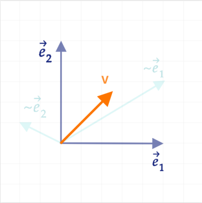
Recall: Tensors

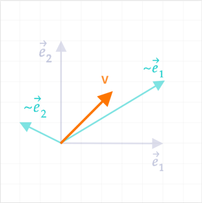
Tensors

"Tensors are mathematical objects that are invariant under a change of coordinates & have components that change in predictable ways."

Jesus Najera







Originally Published:
https://www.setzeus.com/public-blog-post/a-light-intro-to-tensors

Idea of Coordinate Systems

- Vectors and scalars do **not** change if you change the coordinate system.
  - For instance, the temperature is the same if you measure it in Celsius or in Fahrenheit.
  - The size of a room is the same in inches or meters.
- Coordinate system defines **how you look** at the physical quantity.
  - For instance, looking at a room from the door or from the ceiling - the room is still the same but the observation point (origin) is changed.

Formal Definition of vector

A quantity $\mathbf{v}$ is a **vector** if its components transform according to

$$v'_i = L_{ij} v_j$$

under a rotation of the coordinate axes.

Formal Definition of scalar

A quantity s is a **scalar** if it is unchanged by a transformation. That is, if

$$s' = s.$$

Example 1

Example 1

Let us prove that the dot product is indeed a scalar. We must show

$$(\mathbf{a}\cdot\mathbf{b})' = \mathbf{a}\cdot\mathbf{b} \quad \text{that is } (a_i b_i)' = a_i b_i.$$

As **a** and **b** are vectors, their components transform according to

$$a_i' = L_{ij} a_j, \quad \text{and } b_i' = L_{ij} b_j.$$

Thus,

$$\begin{aligned}
(\mathbf{a}\cdot\mathbf{b})' &= a_i' b_i' = L_{ij} a_j L_{ik} b_k && \text{(By definition.)} \\
&= L_{ij} L_{ik} a_j b_k && \text{(Reordering)} \\
&= \delta_{jk} a_j b_k && (L_{ij} L_{ik} = \delta_{jk}) \\
&= a_k b_k && (\delta_{jk} a_j = a_k) \\
&= \mathbf{a}\cdot\mathbf{b} && \text{Thus it is a scalar!}
\end{aligned}$$

Example 2

Suppose that f is a scalar field. We will use the **formal definition** to show that ∇f is a vector.

That is, let us show that

$$(\nabla f)'_i = L_{ij}(\nabla f)_j.$$

Recall $\nabla f = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}\right)$ or equivalently $(\nabla f)_i = \frac{\partial f}{\partial x_i}$.

Since f is a scalar, it holds $f = f'$. Then

$$(\nabla f)'_i = \frac{\partial f'}{\partial x'_i} = \frac{\partial f}{\partial x'_i} = \frac{\partial f}{\partial x_j}\frac{\partial x_j}{\partial x'_i}. \qquad \text{(chain rule)}$$

Example 2 - Part 2

Example 2

So far, we have

$$(\nabla f)'_i = \frac{\partial f}{\partial x_j}\frac{\partial x_j}{\partial x'_i}.$$

We know that

$$\frac{\partial x'_i}{\partial x_j} = L_{ij} \qquad \text{and} \qquad \frac{\partial x_i}{\partial x'_j} = L_{ji}.$$

Thus,

$$(\nabla f)'_i = L_{ij}\frac{\partial f}{\partial x_j} = L_{ij}(\nabla f)_j.$$

We conclude that $(\nabla f)'_i = L_{ij}(\nabla f)_j$, thus ∇f is a **vector**.

Task

Practical question: Your turn!

Show that ∇ is a vector using the **transformation law**.

Practical question: Your turn!

Let f be a scalar field. Show that $\nabla \cdot (\nabla f)$ is a scalar using the **transformation law**.

Your turn!

Use the **transformation law** and the fact that ∇ is a vector to show:

If $\mathbf{u}$ is a vector field, then $\nabla \cdot \mathbf{u}$ is a scalar field.

Next Lecture

Next time...

- Chapter 3: Local Coordinate Transform.
  - Dual bases.