Slides week 22
Tensor Analysis

Lecture 1

Paula Lins

University of Lincoln

2025/26

Chapter 3: Local Coordinate Transform

Today: Chapter 3–Local Coordinate Transform

1. Preliminaries,
2. Dual bases,
3. Covariant and contravariant components of a vector,
4. The transformation rule,
5. The relationship between covartiant and contravariant components, and
6. Arc length and the metric tensor.

Reminder

Coordinate Systems

Definition of Coordinate Systems

Coordinate systems are defined by a set of **basis vectors**.

Definitions

- A coordinate system is **orthogonal** if its basis vectors intersect at 90° angles, i.e.,

$$\mathbf{v}_i \cdot \mathbf{v}_j = 0, \text{ for } i \neq j.$$

- A coordinate system is **orthonormal** if it is orthogonal and its basis vectors have magnitude 1, i.e.,
  - $\mathbf{v}_i \cdot \mathbf{v}_j = 0$, for $i \neq j$, and
  - $|\mathbf{v}_i| = \sqrt{\mathbf{v}_i \cdot \mathbf{v}_i} = 1$.

Cartesian and generalised coordinate systems

Cartesian coordinate system

The **Cartesian coordinate** system is an example of **orthonormal coordinate system.**

$$\mathbf{i} = (1, 0, 0), \quad \mathbf{j} = (0, 1, 0), \quad \mathbf{k} = (0, 0, 1)$$

Definition

Generalised coordinate systems do not necessarily have orthogonal coordinate curves.

Transformations in 3D

Let $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ be basis vectors. Consider the position vector

$$\mathbf{r} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3 = \mathbf{e}_j x_j.$$

Rotating the basis vectors, we obtain new basis vectors $\mathbf{e}'_i$, and

$$x'_i = \mathbf{e}'_i \cdot \mathbf{r} = \mathbf{e}'_i \cdot \mathbf{e}_j x_j = L_{ij} x_j.$$

Where L_{ij} is the **transformation matrix**.

- L is **orthogonal** i.e. $L^T L = I = LL^T$. In particular,

$$L_{ij} L_{kj} = \delta_{ik} = L_{ji} L_{jk}.$$

- We have shown that

$$\frac{\partial x'_i}{\partial x_j} = L_{ij} \qquad \text{and} \qquad \frac{\partial x_i}{\partial x'_j} = L_{ji}.$$

Formal Definition of vector

A quantity $\mathbf{v}$ is a **vector** if its components transform according to

$$v_i' = L_{ij} v_j$$

under a rotation of the coordinate axes.

Formal Definition of scalar

A quantity s is a **scalar** if it is unchanged by a transformation. That is, if

$$s' = s.$$

Dual Basis

Dual Bases

Definition.

Two bases $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ and $\mathbf{e}^1$, $\mathbf{e}^2$, $\mathbf{e}^3$ are **dual** if they satisfy

$$\mathbf{e}_i \cdot \mathbf{e}^k = \begin{cases} 0 & \text{if } i \neq k, \\ 1 & \text{if } i = k. \end{cases}$$

In other words, two bases are **dual** if each vector of one basis is **perpendicular** to two vectors of the other basis.

Constructing dual bases

Given a basis $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$, we are going to prove that there exists a dual basis $\mathbf{e}^1$, $\mathbf{e}^2$, $\mathbf{e}^3$.

This will be a **constructive proof**, meaning we are going to construct this dual basis.

- First, $\mathbf{e}^1$ must be perpendicular to the vectors $\mathbf{e}_2$ and $\mathbf{e}_3$. Thus,

$$\mathbf{e}^1 = m(\mathbf{e}_2 \times \mathbf{e}_3).$$

- Now, we need $\mathbf{e}_1 \cdot \mathbf{e}^1 = 1$, so that

$$m(\mathbf{e}_1 \cdot (\mathbf{e}_2 \times \mathbf{e}_3)) = 1.$$

  That is

$$m = \frac{1}{\mathbf{e}_1 \cdot (\mathbf{e}_2 \times \mathbf{e}_3)} =: \frac{1}{V}.$$

Constructing dual bases

So far, we have

- $\mathbf{e}^1 = m(\mathbf{e}_2 \times \mathbf{e}_3)$, and
- $$m = \frac{1}{\mathbf{e}_1 \cdot (\mathbf{e}_2 \times \mathbf{e}_3)} =: \frac{1}{V}.$$

Here, $|V|$ denotes the volume of the **parallelepiped** spanned by the basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$.

We conclude that

$$\mathbf{e}^1 = \frac{\mathbf{e}_2 \times \mathbf{e}_3}{\mathbf{e}_1 \cdot (\mathbf{e}_2 \times \mathbf{e}_3)} = \frac{\mathbf{e}_2 \times \mathbf{e}_3}{V}.$$

Constructing dual bases

Similarly, we find $\mathbf{e}^2$ and $\mathbf{e}^3$. The dual basis is given by

$$\mathbf{e}^1 = \frac{\mathbf{e}_2 \times \mathbf{e}_3}{\mathbf{e}_1 \cdot (\mathbf{e}_2 \times \mathbf{e}_3)}, \qquad \mathbf{e}^2 = \frac{\mathbf{e}_3 \times \mathbf{e}_1}{\mathbf{e}_2 \cdot (\mathbf{e}_3 \times \mathbf{e}_1)}, \qquad \mathbf{e}^3 = \frac{\mathbf{e}_1 \times \mathbf{e}_2}{\mathbf{e}_3 \cdot (\mathbf{e}_1 \times \mathbf{e}_2)}.$$

We obtain the formula

$$\mathbf{e}^i = \frac{\mathbf{e}_j \times \mathbf{e}_k}{\mathbf{e}_i \cdot (\mathbf{e}_j \times \mathbf{e}_k)} = \frac{\mathbf{e}_j \times \mathbf{e}_k}{V},$$

where (i, j, k) is a cyclic permutation of $(1, 2, 3)$.

Starting from the dual basis

In the same way, we can show

$$\mathbf{e}_1 = \frac{\mathbf{e}^2 \times \mathbf{e}^3}{\mathbf{e}^1 \cdot (\mathbf{e}^2 \times \mathbf{e}^3)}, \qquad \mathbf{e}_2 = \frac{\mathbf{e}^3 \times \mathbf{e}^1}{\mathbf{e}^2 \cdot (\mathbf{e}^3 \times \mathbf{e}^1)}, \qquad \mathbf{e}_3 = \frac{\mathbf{e}^1 \times \mathbf{e}^2}{\mathbf{e}^3 \cdot (\mathbf{e}^1 \times \mathbf{e}^2)}.$$

Thus, we get the formula

$$\mathbf{e}_i = \frac{\mathbf{e}^j \times \mathbf{e}^k}{\mathbf{e}^i \cdot (\mathbf{e}^j \times \mathbf{e}^k)} = \frac{\mathbf{e}^j \times \mathbf{e}^k}{V'},$$

where (i, j, k) is a cyclic permutation of $(1, 2, 3)$, and

$|V'|$ denotes the volume of the **parallelepiped** spanned by the basis $\mathbf{e}^1, \mathbf{e}^2, \mathbf{e}^3$.

Example.

Given the basis

$$\mathbf{e}_1 = 2\mathbf{i}_1 + \mathbf{i}_3, \quad \mathbf{e}_2 = \mathbf{i}_1 + 3\mathbf{i}_2, \quad \mathbf{e}_3 = 4\mathbf{i}_3,$$

where $\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3$ is an orthonormal basis, find the dual basis $\mathbf{e}^1, \mathbf{e}^2, \mathbf{e}^3$.

Solution

We have

$$\mathbf{e}_1 = 2\mathbf{i}_1 + \mathbf{i}_3, \quad \mathbf{e}_2 = \mathbf{i}_1 + 3\mathbf{i}_2, \quad \mathbf{e}_3 = 4\mathbf{i}_3.$$

For simplicity, we write

$$\mathbf{e}_1 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, \qquad \mathbf{e}_2 = \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}, \qquad \mathbf{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix}.$$

Solution – Part 2

Recall that the dual basis is given by the formula

$$\mathbf{e}^i = \frac{\mathbf{e}_j \times \mathbf{e}_k}{\mathbf{e}_i \cdot (\mathbf{e}_j \times \mathbf{e}_k)}.$$

Thus, we need the cross products $\mathbf{e}_1 \times \mathbf{e}_2$, $\mathbf{e}_2 \times \mathbf{e}_3$ and $\mathbf{e}_3 \times \mathbf{e}_1$.

$$\mathbf{e}_1 \times \mathbf{e}_2 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \times \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} = \begin{pmatrix} -3 \\ 1 \\ 6 \end{pmatrix}$$

$$\mathbf{e}_2 \times \mathbf{e}_3 = \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix} = \begin{pmatrix} 12 \\ -4 \\ 0 \end{pmatrix}$$

$$\mathbf{e}_3 \times \mathbf{e}_1 = \begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix} \times \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 8 \\ 0 \end{pmatrix}.$$

Solution – Part 3

Again,

$$\mathbf{e}^i = \frac{\mathbf{e}_j \times \mathbf{e}_k}{\mathbf{e}_i \cdot (\mathbf{e}_j \times \mathbf{e}_k)}.$$

We already computed the cross products, so we still need to find V. We have

$$V = \mathbf{e}_1 \cdot (\mathbf{e}_2 \times \mathbf{e}_3) = \mathbf{e}_2 \cdot (\mathbf{e}_3 \times \mathbf{e}_1) = \mathbf{e}_3 \cdot (\mathbf{e}_1 \times \mathbf{e}_2).$$

Let us pick one and compute. We know

$$\mathbf{e}_1 = 2\mathbf{i}_1 + \mathbf{i}_3 \quad \text{and} \quad \mathbf{e}_2 \times \mathbf{e}_3 = 12\mathbf{i}_1 - 4\mathbf{i}_2.$$

Thus

$$V = \mathbf{e}_1 \cdot (\mathbf{e}_2 \times \mathbf{e}_3) = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 12 \\ -4 \\ 0 \end{pmatrix} = 24.$$

Solution – Part 4

Now we can construct the dual basis

$$\mathbf{e}^1 = \frac{\mathbf{e}_2 \times \mathbf{e}_3}{\mathbf{e}_1 \cdot (\mathbf{e}_2 \times \mathbf{e}_3)} = \frac{1}{24}\begin{pmatrix} 12 \\ -4 \\ 0 \end{pmatrix} = \begin{pmatrix} 1/2 \\ -1/6 \\ 0 \end{pmatrix},$$

$$\mathbf{e}^2 = \frac{\mathbf{e}_3 \times \mathbf{e}_1}{\mathbf{e}_2 \cdot (\mathbf{e}_3 \times \mathbf{e}_1)} = \frac{1}{24}\begin{pmatrix} 0 \\ 8 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1/3 \\ 0 \end{pmatrix},$$

$$\mathbf{e}^3 = \frac{\mathbf{e}_1 \times \mathbf{e}_2}{\mathbf{e}_3 \cdot (\mathbf{e}_1 \times \mathbf{e}_2)} = \frac{1}{24}\begin{pmatrix} -3 \\ 1 \\ 6 \end{pmatrix} = \begin{pmatrix} -1/8 \\ 1/24 \\ 1/4 \end{pmatrix}.$$

Practical Question

Your turn!

Let $\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3$ be the standard orthonormal basis of $\mathbb{R}^3$. Consider the following basis

$$\mathbf{e}_1 = \mathbf{i}_1 + 2\mathbf{i}_2 + 4\mathbf{i}_3, \quad \mathbf{e}_2 = \mathbf{i}_2, \quad \mathbf{e}_3 = \mathbf{i}_1 + 2\mathbf{i}_2 + 5\mathbf{i}_3.$$

Find the dual basis $\mathbf{e}^1, \mathbf{e}^2, \mathbf{e}^3$.

Example: The dual basis of an orthonormal basis

Let $\mathbf{e}_1 = \mathbf{i}_1, \quad \mathbf{e}_2 = \mathbf{i}_2, \quad \mathbf{e}_3 = \mathbf{i}_3$ be an orthornormal basis.

Then its dual basis is itself an orthonormal basis, consisting of the same vectors:

$$\mathbf{e}^1 = \frac{\mathbf{e}_2 \times \mathbf{e}_3}{\mathbf{e}_1 \cdot (\mathbf{e}_2 \times \mathbf{e}_3)} = \frac{\mathbf{i}_2 \times \mathbf{i}_3}{\mathbf{i}_1 \cdot (\mathbf{i}_2 \times \mathbf{i}_3)} = \frac{\mathbf{i}_1}{\mathbf{i}_1 \cdot \mathbf{i}_1} = \frac{\mathbf{i}_1}{1} = \mathbf{i}_1,$$

and similarly

$$\mathbf{e}^2 = \mathbf{i}_2, \qquad \mathbf{e}^3 = \mathbf{i}_3.$$

Summary: Dual Basis

The dual basis $\mathbf{e}^i$ can be found from the original basis $\mathbf{e}_i$ via

$$\mathbf{e}^i = \frac{\mathbf{e}_j \times \mathbf{e}_k}{\mathbf{e}_i \cdot (\mathbf{e}_j \times \mathbf{e}_k)},$$

and the original basis $\mathbf{e}_i$ can be found from the dual basis $\mathbf{e}^i$ via

$$\mathbf{e}_i = \frac{\mathbf{e}^j \times \mathbf{e}^k}{\mathbf{e}^i \cdot (\mathbf{e}^j \times \mathbf{e}^k)},$$

where (i, j, k) is a cyclic permutation of $(1, 2, 3)$.

Covariant and contravariant components of a vector

Vector expansion

Vector expansion

Let $\mathbf{e}_1$,$\mathbf{e}_2$, $\mathbf{e}_3$ be any basis of $\mathbb{R}^3$. Then, by definition of basis, any vector $\mathbf{v} = (v_1, v_2, v_3)$ can be expressed as a linear combination of $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$.

Example

The vector $\mathbf{v} = (1, 0, 1)$ can be expressed as

- $$\mathbf{v} = 1 \cdot \mathbf{i} + 0 \cdot \mathbf{j} + 1 \cdot \mathbf{k}$$

  in the basis $\mathbf{i} = (1, 0, 0)$, $\mathbf{j} = (0, 1, 0)$, $\mathbf{k} = (0, 0, 1)$.

- $$\mathbf{v} = 1 \cdot \mathbf{e}_1 - 1 \cdot \mathbf{e}_2 + 1 \cdot \mathbf{e}_3$$

  in the basis $\mathbf{e}_1 = (1, 0, 0)$, $\mathbf{e}_2 = (1, 1, 0)$, $\mathbf{e}_3 = (1, 1, 1)$.

Expanding vectors

Again, given **any** basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$, can express any vector $\mathbf{A}$ in terms of this basis:

$$\mathbf{A} = A^1\mathbf{e}_1 + A^2\mathbf{e}_2 + A^3\mathbf{e}_3,$$

where A^1, A^2, A^3 are scalars (in our context, real numbers).

The scalars A^1, A^2, A^3 are known as the **expansion coefficients**.

In the following slides, we will learn how to determine the exact values of A^1, A^2, A^3. We will explore the following scenarios:

- The basis is **orthonormal**.
- The basis is **orthogonal**.
- The basis is **generalised**.

Vectors in terms of orthonormal bases

Vectors in terms of orthonormal bases

Let $\mathbf{i}_1$,$\mathbf{i}_2$, $\mathbf{i}_3$ be an **orthonormal** basis.

Let the expansion of $\mathbf{A}$ be given by

$$\mathbf{A} = A^1\mathbf{i}_1 + A^2\mathbf{i}_2 + A^3\mathbf{i}_3 = A^k\mathbf{i}_k.$$

We can determine the precise value of A^1, A^2, A^3 as follows: since $\mathbf{i}_1$,$\mathbf{i}_2$, $\mathbf{i}_3$ are orthonormal, we have $\mathbf{i}_k \cdot \mathbf{i}_j = \delta_{kj}$. Thus,

$$\mathbf{A} \cdot \mathbf{i}_j = (A^k\mathbf{i}_k) \cdot \mathbf{i}_j = A^k(\mathbf{i}_k \cdot \mathbf{i}_j) = A^k\delta_{kj} = A^j.$$

Consequently,

$$\mathbf{A} = (\mathbf{A} \cdot \mathbf{i}_1)\mathbf{i}_1 + (\mathbf{A} \cdot \mathbf{i}_2)\mathbf{i}_2 + (\mathbf{A} \cdot \mathbf{i}_3)\mathbf{i}_3.$$

Vectors in terms of orthogonal bases

Previous slide: If $\mathbf{i}_1$,$\mathbf{i}_2$, $\mathbf{i}_3$ is an **orthonormal** basis, any vector $\mathbf{A}$ can be described by:

$$\mathbf{A} = (\mathbf{A}\cdot\mathbf{i}_1)\mathbf{i}_1 + (\mathbf{A}\cdot\mathbf{i}_2)\mathbf{i}_2 + (\mathbf{A}\cdot\mathbf{i}_3)\mathbf{i}_3.$$

Let $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ be an **orthogonal** basis.

Although $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ might not be orthonormal, the following vectors are

$$\mathbf{i}_1 = \frac{\mathbf{e}_1}{e_1}, \quad \mathbf{i}_2 = \frac{\mathbf{e}_2}{e_2}, \quad \mathbf{i}_3 = \frac{\mathbf{e}_3}{e_3} \quad (e_i = |\mathbf{e}_i| \text{ for } i = 1, 2, 3)$$

Then, by substitution

$$\mathbf{A} = \frac{\mathbf{A}\cdot\mathbf{e}_1}{e_1^2}\mathbf{e}_1 + \frac{\mathbf{A}\cdot\mathbf{e}_2}{e_2^2}\mathbf{e}_2 + \frac{\mathbf{A}\cdot\mathbf{e}_3}{e_3^2}\mathbf{e}_3.$$

Vectors in generalised coordinate systems

To expand a vector in terms of a basis that is **not** orthogonal, we use the so-called **dual basis method**.

Recall: Dual basis

Two bases $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ and $\mathbf{e}^1$, $\mathbf{e}^2$, $\mathbf{e}^3$ are **dual** if they satisfy

$$\mathbf{e}_i \cdot \mathbf{e}^k = \begin{cases} 0 & \text{if } i \neq k, \\ 1 & \text{if } i = k. \end{cases}$$

In other words, two bases are **dual** if each vector of one basis is **perpendicular** to two vectors of the other basis.

Expansion coefficients

Expansion coefficients in generalised coordinate systems

Let $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ and $\mathbf{e}^1$, $\mathbf{e}^2$, $\mathbf{e}^3$ be dual bases.

By definition of basis, any vector $\mathbf{A}$ can be expanded with respect to each of these bases:

1. Expansion with respect to the basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$

$$\mathbf{A} = A^1\mathbf{e}_1 + A^2\mathbf{e}_2 + A^3\mathbf{e}_3 = A^i\mathbf{e}_i.$$

2. Expansion with respect to the dual basis $\mathbf{e}^1, \mathbf{e}^2, \mathbf{e}^3$

$$\mathbf{A} = A_1\mathbf{e}^1 + A_2\mathbf{e}^2 + A_3\mathbf{e}^3 = A_i\mathbf{e}^i.$$

The A^i and A_i are the **expansion coefficients** of $\mathbf{A}$.

Contravariant component

Let $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ and $\mathbf{e}^1$, $\mathbf{e}^2$, $\mathbf{e}^3$ be dual bases.

By definition of basis, any vector $\mathbf{A}$ can be expanded with respect to each of these bases:

1. Expansion with respect to the basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$

$$\mathbf{A} = A^1\mathbf{e}_1 + A^2\mathbf{e}_2 + A^3\mathbf{e}_3 = A^i\mathbf{e}_i.$$

2. Expansion with respect to the dual basis $\mathbf{e}^1, \mathbf{e}^2, \mathbf{e}^3$

$$\mathbf{A} = A_1\mathbf{e}^1 + A_2\mathbf{e}^2 + A_3\mathbf{e}^3 = A_i\mathbf{e}^i.$$

The A^i are called the **contravariant** components of $\mathbf{A}$.

Covariant component

Let $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ and $\mathbf{e}^1$, $\mathbf{e}^2$, $\mathbf{e}^3$ be dual bases.

By definition of basis, any vector $\mathbf{A}$ can be expanded with respect to each of these bases:

1. Expansion with respect to the basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$

$$\mathbf{A} = A^1\mathbf{e}_1 + A^2\mathbf{e}_2 + A^3\mathbf{e}_3 = A^i\mathbf{e}_i.$$

2. Expansion with respect to the dual basis $\mathbf{e}^1, \mathbf{e}^2, \mathbf{e}^3$

$$\mathbf{A} = A_1\mathbf{e}^1 + A_2\mathbf{e}^2 + A_3\mathbf{e}^3 = A_i\mathbf{e}^i.$$

The A_i are called the **covariant** components of $\mathbf{A}$.

Finding the contravariant and covariant coefficients

Finding the contravariant and covariant coefficients

We will demonstrate:

- how to compute the **contravariant components** A^1, A^2, A^3 in terms of the dual basis, and

- how to compute the **covariant components** A_1, A_2, A_3 in terms of the original basis.

Finding contravariant components

We want to find a formula for the **contravariant components** A^1, A^2, A^3 of a vector $\mathbf{A}$.

In other words, we need to find the coefficients A^1, A^2, A^3 in the expansion

$$\mathbf{A} = A^1\mathbf{e}_1 + A^2\mathbf{e}_2 + A^3\mathbf{e}_3.$$

Using the identity

$$\mathbf{e}_k \cdot \mathbf{e}^i = \begin{cases} 0 & \text{if } i \neq k, \\ 1 & \text{if } i = k \end{cases} = \delta_{ik}.$$

we obtain for each $i = 1, 2, 3$

$$\mathbf{A} \cdot \mathbf{e}^i = \left(\sum_{k=1}^{3} A^k \mathbf{e}_k \right) \cdot \mathbf{e}^i = \sum_{k=1}^{3} A^k \left(\mathbf{e}_k \cdot \mathbf{e}^i \right) = \sum_{k=1}^{3} A^k \delta_{ik} = A^i.$$

Finding covariant components

We want to find a formula for the **covariant components** A_1, A_2, A_3 of a vector $\mathbf{A}$.

In other words, we need to find the coefficients A_1, A_2, A_3 in the expansion

$$\mathbf{A} = A_1\mathbf{e}^1 + A_2\mathbf{e}^2 + A_3\mathbf{e}^3.$$

Using the identity

$$\mathbf{e}^k \cdot \mathbf{e}_i = \begin{cases} 0 & \text{if } i \neq k, \\ 1 & \text{if } i = k \end{cases} = \delta_{ik},$$

we obtain for each $i = 1, 2, 3$

$$\mathbf{A} \cdot \mathbf{e}_i = \left(\sum_{k=1}^{3} A_k \mathbf{e}^k \right) \cdot \mathbf{e}_i = \sum_{k=1}^{3} A_k \left(\mathbf{e}^k \cdot \mathbf{e}_i \right) = \sum_{k=1}^{3} A_k \delta_{ik} = A_i.$$

Summary: Covariant and contravariant components

Summary: Covariant and contravariant components

Using dual bases, we have found that a vector $\mathbf{A}$ can be expanded

- with respect to the vectors of one basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ via

$$\mathbf{A} = A^1\mathbf{e}_1 + A^2\mathbf{e}_2 + A^3\mathbf{e}_3 = A^i\mathbf{e}_i.$$

- with respect to the vectors of the dual basis $\mathbf{e}^1, \mathbf{e}^2, \mathbf{e}^3$ via

$$\mathbf{A} = A_1\mathbf{e}^1 + A_2\mathbf{e}^2 + A_3\mathbf{e}^3 = A_i\mathbf{e}^i,$$

where

- $A^i = \mathbf{A} \cdot \mathbf{e}^i$ are called the **contravariant** components of $\mathbf{A}$, and
- $A_i = \mathbf{A} \cdot \mathbf{e}_i$ are called the **covariant** components of $\mathbf{A}$.

Example

In the previous example, we have found that basis

$$\mathbf{e}_1 = 2\mathbf{i}_1 + \mathbf{i}_3, \quad \mathbf{e}_2 = \mathbf{i}_1 + 3\mathbf{i}_2, \quad \mathbf{e}_3 = 4\mathbf{i}_3,$$

has dual basis

$$\mathbf{e}^1 = \begin{pmatrix} 1/2 \\ -1/6 \\ 0 \end{pmatrix}, \qquad \mathbf{e}^2 = \begin{pmatrix} 0 \\ 1/3 \\ 0 \end{pmatrix}, \qquad \mathbf{e}^3 = \begin{pmatrix} -1/8 \\ 1/24 \\ 1/4 \end{pmatrix}.$$

Find the covariant and contravariant components of the vector joining the origin to the point $(8, 0, 1)$.

Example: Covariant components

Finding the **covariant** components of $\mathbf{J} = (8, 0, 1)^T$: The basis is

$$\mathbf{e}_1 = (2, 0, 1)^T, \quad \mathbf{e}_2 = (1, 3, 0)^T, \quad \mathbf{e}_3 = (0, 0, 4)^T.$$

The **covariant** components J_1, J_2, J_3 are such that

$$\mathbf{J} = J_1\mathbf{e}^1 + J_2\mathbf{e}^2 + J_3\mathbf{e}^3 \quad \text{with } J_i = \mathbf{J} \cdot \mathbf{e}_i$$

Thus, we have

$$J_1 = \mathbf{J} \cdot \mathbf{e}_1 = \begin{pmatrix} 8 \\ 0 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} = 20, \qquad J_2 = \mathbf{J} \cdot \mathbf{e}_2 = \begin{pmatrix} 8 \\ 0 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} = 8,$$

$$J_3 = \mathbf{J} \cdot \mathbf{e}_3 = \begin{pmatrix} 8 \\ 0 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix}. = 16.$$

Example: Contravariant components

Finding the **contravariant** components of $\mathbf{J} = (8, 0, 4)^T$:

$$\mathbf{e}^1 = (1/2, -1/6, 0)^T, \quad \mathbf{e}^2 = (0, 1/3, 0)^T, \quad \mathbf{e}^3 = (-1/8, 1/24, 1/4)^T$$

The **contravariant** components J^1, J^2, J^3 are such that

$$\mathbf{J} = J^1\mathbf{e}_1 + J^2\mathbf{e}_2 + J^3\mathbf{e}_3 \quad \text{with } J^i = \mathbf{J} \cdot \mathbf{e}^i.$$

Thus, we have

$$J^1 = \mathbf{J} \cdot \mathbf{e}^1 = \begin{pmatrix} 8 \\ 0 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{6} \\ 0 \end{pmatrix} = 4, \quad J^2 = \mathbf{J} \cdot \mathbf{e}^2 = \begin{pmatrix} 8 \\ 0 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ \frac{1}{3} \\ 0 \end{pmatrix} = 0,$$

$$J^3 = \mathbf{J} \cdot \mathbf{e}^3 = \begin{pmatrix} 8 \\ 0 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} -\frac{1}{8} \\ \frac{1}{24} \\ \frac{1}{4} \end{pmatrix} = 0.$$

Your turn!

In the previous exercise, we have found that the basis

$$\mathbf{e}_1 = \mathbf{i}_1 + 2\mathbf{i}_2 + 4\mathbf{i}_3, \quad \mathbf{e}_2 = \mathbf{i}_2, \quad \mathbf{e}_3 = 3\mathbf{i}_1 + 2\mathbf{i}_2 + 5\mathbf{i}_3,$$

has dual basis

$$\mathbf{e}^1 = \begin{pmatrix} 5 \\ 0 \\ -1 \end{pmatrix}, \qquad \mathbf{e}^2 = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \qquad \mathbf{e}^3 = \begin{pmatrix} -4 \\ 0 \\ 1 \end{pmatrix}.$$

Find the covariant and contravariant components of the vector joining the origin to the point $(1, 1, 1)$.

Next Lecture

Next time...

- Chapter 3: Local Coordinate Transform.
  - The transformation rule,
  - The relationship between covariant and contravariant components.

Final answer of the task

Answer

Check if you got it right!

The **covariant** components of $\mathbf{J} = (1, 1, 1)^T$ are

$$J_1 = 7, \qquad J_2 = 1, \qquad J_3 = 8.$$

The **contravariant** components of $\mathbf{J} = (1, 1, 1)^T$ are

$$J^1 = 4, \qquad J^2 = -1, \qquad J^3 = -3.$$