Tensor Analysis

Slides week 22 – Lecture 2

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2025/26

Chapter 3: Local Coordinate Transform

Today: Chapter 3–Local Coordinate Transform

1. Preliminaries,
2. Dual bases,
3. Covariant and contravariant components of a vector,
4. The transformation rule,
5. The relationship between covariant and contravariant components, and
6. Arc length and the metric tensor.

The transformation rule

Recall: Covariant and contravariant components

Covariant and contravariant components

Using dual bases, we have found that a vector $\mathbf{A}$ can be expanded

- with respect to the vectors of one basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ via

$$\mathbf{A} = A^1\mathbf{e}_1 + A^2\mathbf{e}_2 + A^3\mathbf{e}_3 = A^i\mathbf{e}_i.$$

- with respect to the vectors of the dual basis $\mathbf{e}^1, \mathbf{e}^2, \mathbf{e}^3$ via

$$\mathbf{A} = A_1\mathbf{e}^1 + A_2\mathbf{e}^2 + A_3\mathbf{e}^3 = A_i\mathbf{e}^i.$$

Recall that

- $A^i = \mathbf{A} \cdot \mathbf{e}^i$ are called the **contravariant** components of $\mathbf{A}$, and
- $A_i = \mathbf{A} \cdot \mathbf{e}_i$ are called the **covariant** components of $\mathbf{A}$.

Coordinate systems

Let $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ and $\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3$ be bases of two coordinate systems.

- In the system defined by $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$, let $\mathbf{A}$ have covariant components A_i and contravariant components A^i, that is

$$\mathbf{A} = A^i \mathbf{e}_i = A_i \mathbf{e}^i.$$

- In the system defined by $\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3$, let $\mathbf{A}$ have covariant components A'_i and contravariant components A'^i, that is

$$\mathbf{A} = A'^i \mathbf{e}_i = A'_i \mathbf{e}^i.$$

How do we transform the components from one coordinate system to another?

Coefficients of expansions

First, let $L^1_{i'}, L^2_{i'}, L^3_{i'}$ be the coefficients of the expansion of $\mathbf{e}'_i$ with respect to the vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$, that is

$$\begin{aligned}
\mathbf{e}'_1 &= L^1_{1'}\mathbf{e}_1 + L^2_{1'}\mathbf{e}_2 + L^3_{1'}\mathbf{e}_3 \\
\mathbf{e}'_2 &= L^1_{2'}\mathbf{e}_1 + L^2_{2'}\mathbf{e}_2 + L^3_{2'}\mathbf{e}_3 \\
\mathbf{e}'_3 &= L^1_{3'}\mathbf{e}_1 + L^2_{3'}\mathbf{e}_2 + L^3_{3'}\mathbf{e}_3
\end{aligned}$$

or more concisely

$$\mathbf{e}'_i = L^j_{i'}\mathbf{e}_j.$$

Transforming between coordinate systems - Coefficients of expansions - Part 2

Coefficients of expansions

Let us find an expression for $L^{j}_{i'}$. Here, we are working in **vector notation**. Since

$$\mathbf{e}'_i = \sum_{k=1}^{3} L^{k}_{i'}\mathbf{e}_k \quad \text{and} \quad \mathbf{e}_k \cdot \mathbf{e}^j = \delta_{ik}, \qquad \text{(dual basis)}$$

we see that

$$\mathbf{e}'_i \cdot \mathbf{e}^j = \left(\sum_{k=1}^{3} L^{k}_{i'}\mathbf{e}_k\right) \cdot \mathbf{e}^j = \sum_{k=1}^{3} L^{k}_{i'}\left(\mathbf{e}_k \cdot \mathbf{e}^j\right) = \sum_{k=1}^{3} L^{k}_{i'}\delta_{kj} = L^{j}_{i'}.$$

Similarly, we can find the inverse transformation

$$L^{j'}_{i} = \mathbf{e}_i \cdot \mathbf{e}'^j.$$

Transformation rule for the covariant components

Transformation rule for the covariant components

If we start with an expansion $\mathbf{A} = A_j \mathbf{e}^j$, and take the dot product with $\mathbf{e}'_i$, we get

$$\begin{aligned} \mathbf{A} \cdot \mathbf{e}'_i &= \left(A_j \mathbf{e}^j\right) \cdot \mathbf{e}'_i \\ &= A_j (\mathbf{e}'{}_i \cdot \mathbf{e}^j) \\ &= L^j_{i'} A_j \qquad\qquad (L^j_{i'} = \mathbf{e}'_i \cdot \mathbf{e}^j) \end{aligned}$$

In other words, the covariant component A'_i is given by

$$A'_i = \mathbf{A} \cdot \mathbf{e}'_i = L^j_{i'} A_j .$$

This is the **transformation rule** for the **covariant** components.

Transformation rule for the contravariant components

Transformation rule for the contravariant components

Similarly, if we start with an expansion $\mathbf{A} = A^j \mathbf{e}_j$, and take the dot product with $\mathbf{e}'^i$, we get

$$\begin{aligned} \mathbf{A} \cdot \mathbf{e}'^i &= \left(A^j \mathbf{e}'_j\right) \cdot \mathbf{e}^i \\ &= A^j (\mathbf{e}^i \cdot \mathbf{e}'_j) \\ &= L^i_{j'} A^j \qquad\qquad (L^i_{j'} = \mathbf{e}^i \cdot \mathbf{e}'{}_j) \end{aligned}$$

In other words, the contravariant component A'^i is given by

$$A'^i = \mathbf{A} \cdot \mathbf{e}'^i = L^i_{j'} A^j.$$

This is the **transformation rule** for the **contravariant** components.

The transformation rule

The **transformation rule** for the **covariant** components is

$$A'_i = L^j_{i'} A_j.$$

The **transformation rule** for the **contravariant** components is

$$A'^i = L^{i'}_j A^j.$$

Task

Verify the inverse formulas

$$A_i = L^{j'}_i A'_j, \qquad A^i = L^i_{j'} A'^j.$$

Example

Let K be a coordinate system with **orthonormal** basis $\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3$.

Let K' be a new coordinate system with basis vectors

$$\mathbf{e}'_1 = \mathbf{i}_1, \qquad \mathbf{e}'_2 = 2\mathbf{i}_2, \qquad \mathbf{e}'_3 = \mathbf{i}_1 + \mathbf{i}_3.$$

Let us find the expansion coefficients $L^1_{i'}, L^2_{i'}, L^3_{i'}$ for each $i = 1, 2, 3$.

We have $L^n_{m'} = \mathbf{e}'_m \cdot \mathbf{i}_n$, thus

$$L^1_{1'} = \mathbf{e}'_1 \cdot \mathbf{i}_1 = \mathbf{i}_1 \cdot \mathbf{i}_1 = 1,$$
$$L^1_{2'} = \mathbf{e}'_2 \cdot \mathbf{i}_1 = 2\mathbf{i}_2 \cdot \mathbf{i}_1 = 0.$$

Example

Let K be a coordinate system with **orthonormal** basis $\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3$.

Let K' be a new coordinate system with basis vectors

$$\mathbf{e}'_1 = \mathbf{i}_1, \qquad \mathbf{e}'_2 = 2\mathbf{i}_2, \qquad \mathbf{e}'_3 = \mathbf{i}_1 + \mathbf{i}_3.$$

Let us find the expansion coefficients $L^1_{i'}, L^2_{i'}, L^3_{i'}$ for each $i = 1, 2, 3$.

We have $L^n_{m'} = \mathbf{e}'_m \cdot \mathbf{i}_n$, thus

$$\begin{aligned} L^1_{1'} &= 1, \quad L^2_{1'} = 0, \quad L^3_{1'} = 0, \\ L^1_{2'} &= 0, \quad L^2_{2'} = 2, \quad L^3_{2'} = 0, \\ L^1_{3'} &= 1, \quad L^2_{3'} = 0, \quad L^3_{3'} = 1. \end{aligned}$$

Example - Part 3

Example

Let K be a coordinate system with **orthonormal** basis $\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3$.

Let K' be a new coordinate system with basis vectors

$$\mathbf{e}'_1 = \mathbf{i}_1, \qquad \mathbf{e}'_2 = 2\mathbf{i}_2, \qquad \mathbf{e}'_3 = \mathbf{i}_1 + \mathbf{i}_3.$$

Consider the vector $\mathbf{B}$ over K with components

$$\mathbf{B} = (1, 1, 0)^T.$$

Compute the covariant components B'_i of $\mathbf{B}$ in the coordinate system K'.

Example

The transformation rule gives $B'_i = L^j_{i'} B_j$. We have shown

$$\begin{aligned} L^1_{1'} &= 1, \quad L^2_{1'} = 0, \quad L^3_{1'} = 0, \\ L^1_{2'} &= 0, \quad L^2_{2'} = 2, \quad L^3_{2'} = 0, \\ L^1_{3'} &= 1, \quad L^2_{3'} = 0, \quad L^3_{3'} = 1. \end{aligned}$$

Thus, because $\mathbf{B} = (1,1,0)^T$ we get

$$\begin{aligned} B'_1 &= L^j_{1'} B_j = L^1_{1'} B_1 + L^2_{1'} B_2 + L^3_{1'} B_3 = 1 + 0 + 0 = 1, \\ B'_2 &= L^j_{2'} B_j = L^1_{2'} B_1 + L^2_{2'} B_2 + L^3_{2'} B_3 = 0 + 2 + 0 = 2, \\ B'_3 &= L^j_{3'} B_j = L^1_{3'} B_1 + L^2_{3'} B_2 + L^3_{3'} B_3 = 1 + 0 + 0 = 1. \end{aligned}$$

Example - Part 5

Example

We can also find the coordinates of $\mathbf{B}'$ using matrices. We have

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix} \qquad \mathbf{B} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

The transformation rule gives $B'_i = L^j_{i'} B_j$. Thus,

$$\mathbf{B}' = L \cdot \mathbf{B} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}.$$

The relationship between covariant and contravariant components

Covariant to contravariant components - Dot product

Recall: Expansions

Recall we have the following two expansions:

$$\mathbf{A} = A^1\mathbf{e}_1 + A^2\mathbf{e}_2 + A^3\mathbf{e}_3 = A^k\mathbf{e}_k$$

$$\mathbf{A} = A_1\mathbf{e}^1 + A_2\mathbf{e}^2 + A_3\mathbf{e}^3 = A_k\mathbf{e}^k$$

How can we express the covariant components in terms of the contravariant components?

First, take the scalar product with $\mathbf{e}_i$ and $\mathbf{e}^i$, respectively:

$$\mathbf{A}\cdot\mathbf{e}_i = A^k(\mathbf{e}_i\cdot\mathbf{e}_k),$$
$$\mathbf{A}\cdot\mathbf{e}^i = A_k(\mathbf{e}^i\cdot\mathbf{e}^k).$$

Covariant to contravariant components - New notation

New notation

We have two expressions:

$$\mathbf{A} \cdot \mathbf{e}_i = A^k(\mathbf{e}_i \cdot \mathbf{e}_k),$$
$$\mathbf{A} \cdot \mathbf{e}^i = A_k(\mathbf{e}^i \cdot \mathbf{e}^k).$$

To simplify notation, we write

$$\mathbf{e}_i \cdot \mathbf{e}_k = g_{ik} = g_{ki},$$
$$\mathbf{e}^i \cdot \mathbf{e}^k = g^{ik} = g^{ki},$$
$$\mathbf{e}_i \cdot \mathbf{e}^k = g_i^{\cdot k} = \begin{cases} 0 & \text{if } i \neq k \\ 1 & \text{if } i = k. \end{cases} \qquad \text{(Dual bases)}$$

Covariant to contravariant components - Formula

Covariant to contravariant components

Thus, we can rewrite the expressions

$$\mathbf{A} \cdot \mathbf{e}_i = A^k(\mathbf{e}_i \cdot \mathbf{e}_k) \quad \text{and} \quad \mathbf{A} \cdot \mathbf{e}^i = A_k(\mathbf{e}^i \cdot \mathbf{e}^k).$$

using this new notation:

$$\mathbf{A} \cdot \mathbf{e}_i = A^k g_{ik}, \quad \text{and} \quad \mathbf{A} \cdot \mathbf{e}^i = A_k g^{ik}.$$

Recall that $A_i = \mathbf{A} \cdot \mathbf{e}_i$ and $A^i = \mathbf{A} \cdot \mathbf{e}^i$. Thus, we conclude

$$A_i = g_{ik} A^k,$$
$$A^i = g^{ik} A_k.$$

Summary: Covariant to contravariant components

Summary: Covariant to contravariant components

We can express the covariant components A_i of a vector $\mathbf{A}$ in terms of is contravariant components A^i (and vice versa) using the formulas

$$A_i = g_{ik} A^k,$$
$$A^i = g^{ik} A_k,$$

where

$$g_{ik} = g_{ki} = \mathbf{e}_i \cdot \mathbf{e}_k,$$
$$g^{ik} = g^{ki} = \mathbf{e}^i \cdot \mathbf{e}^k.$$

Example 1

Express the scalar product of two vectors $\mathbf{A} \cdot \mathbf{B}$ in terms of their covariant and contravariant components.

Using different combinations of the formulae $\mathbf{A} = A^i e_i = A_i e^i$ and $\mathbf{B} = B^k e_k = B_k e^k$, we see that

$$\mathbf{A} \cdot \mathbf{B} = \underbrace{A^i \mathbf{e}_i \cdot B^k \mathbf{e}_k}_{\text{contravariant}} = \underbrace{A_i \mathbf{e}^i \cdot B_k \mathbf{e}^k}_{\text{covariant}} = \underbrace{A_i \mathbf{e}^i \cdot B^k \mathbf{e}_k}_{\text{mix}} = \underbrace{A^i \mathbf{e}_i \cdot B_k \mathbf{e}^k}_{\text{mix}},$$

Hence,

$$\begin{aligned} \mathbf{A} \cdot \mathbf{B} &= A^i \mathbf{e}_i \cdot B^k \mathbf{e}_k = \mathbf{e}_i \cdot \mathbf{e}_k A^i B^k = g_{ik} A^i B^k, \\ &= A_i \mathbf{e}^i \cdot B_k \mathbf{e}^k = \mathbf{e}^i \cdot \mathbf{e}^k A_i B_k = g^{ik} A_i B_k. \end{aligned}$$

Example 1 - g_{ij} - Part 2

Example 1

We have shown

$$\mathbf{A} \cdot \mathbf{B} = g_{ik} A^i B^k, \\ = g^{ik} A_i B_k.$$

Now, since

$$g_{ik} B^k = B_i \quad \text{and} \quad g^{ik} B_k = B^i,$$

we obtain

$$\mathbf{A} \cdot \mathbf{B} = g_{ik} A^i B^k = A^i B_i, \\ = g^{ik} A_i B_k = A_i B^i.$$

Example 2

Express the cosine of the angle between two vectors in terms of their covariant and contravariant components.

Recall that $\cos(\mathbf{A}, \mathbf{B}) = \frac{\mathbf{A}\cdot\mathbf{B}}{|\mathbf{A}||\mathbf{B}|}$. (Practical 1)

We have just found $\mathbf{A} \cdot \mathbf{B} = A^i B_i = A_i B^i$. Then, we have:

$$|\mathbf{A}| = \sqrt{\mathbf{A} \cdot \mathbf{A}} = \sqrt{A_i A^i}, \quad |\mathbf{B}| = \sqrt{\mathbf{B} \cdot \mathbf{B}} = \sqrt{B_i B^i}.$$

We conclude

$$\cos(\mathbf{A}, \mathbf{B}) = \frac{A_i B^i}{\sqrt{A_i A^i}\sqrt{B_i B^i}} = \frac{A^i B_i}{\sqrt{A_i A^i}\sqrt{B_i B^i}}.$$

Example 3

Express the vector product $\mathbf{A} \times \mathbf{B}$ of two vectors in terms of their covariant and contravariant components.

(That is, for $\mathbf{C} = \mathbf{A} \times \mathbf{B} = C_i \mathbf{e}^i$ what are the coefficients C_i?)

Using $\mathbf{A} = A^j e_j$ and $\mathbf{B} = B^k e_k$, we obtain in **vector notation**

$$\mathbf{A} \times \mathbf{B} = \left(\sum_{j=1}^{3} A^j e_j \right) \times \left(\sum_{k=1}^{3} B^k e_k \right)$$

$$= \sum_{j,k=1}^{3} A^j B^k (\mathbf{e}_j \times \mathbf{e}_k)$$

Example 3

Continuing

$$\begin{aligned}\mathbf{A}\times\mathbf{B} &= \sum_{j,k=1}^{3} A^j B^k(\mathbf{e}_j\times\mathbf{e}_k)\\ &= A^1B^2(\mathbf{e}_1\times\mathbf{e}_2)+A^1B^3(\mathbf{e}_1\times\mathbf{e}_3)+A^2B^3(\mathbf{e}_2\times\mathbf{e}_3)\\ &-A^3B^1(\mathbf{e}_1\times\mathbf{e}_3)-A^3B^2(\mathbf{e}_2\times\mathbf{e}_3)-A^2B^1(\mathbf{e}_1\times\mathbf{e}_2)\\ \\ &= (A^1B^2-A^2B^1)(\mathbf{e}_1\times\mathbf{e}_2)+(A^1B^3-A^3B^1)(\mathbf{e}_1\times\mathbf{e}_3)\\ &+(A^2B^3-A^3B^2)(\mathbf{e}_2\times\mathbf{e}_3)\\ \\ &= \sum_{i<j}(A^jB^k-A^kB^j)(\mathbf{e}_j\times\mathbf{e}_k).\end{aligned}$$

Example 3

We have so far
$$\mathbf{A} \times \mathbf{B} = \sum_{i<j} (A^j B^k - A^k B^j)(\mathbf{e}_j \times \mathbf{e}_k).$$

By definition, $V = \mathbf{e}_i \cdot (\mathbf{e}_j \times \mathbf{e}_k)$ is such that

$$\mathbf{e}^i = \frac{\mathbf{e}_j \times \mathbf{e}_k}{V} \quad \text{thus } \mathbf{e}_j \times \mathbf{e}_k = V\mathbf{e}^i,$$

whenever (i, j, k) is a cyclic permutation of $(1, 2, 3)$. As a consequence

$$\begin{aligned}\mathbf{C} = \mathbf{A} \times \mathbf{B} &= (A^1B^2 - A^2B^1)(\mathbf{e}_1 \times \mathbf{e}_2) + (A^1B^3 - A^3B^1)(\mathbf{e}_1 \times \mathbf{e}_3) \\ &+ (A^2B^3 - A^3B^2)(\mathbf{e}_2 \times \mathbf{e}_3) \\ &= (A^jB^k - A^kB^j)V\mathbf{e}^i = C_i\mathbf{e}^i.\end{aligned}$$

Example 3

Consequently

$$\begin{aligned}
\mathbf{C} = \mathbf{A} \times \mathbf{B} &= (A^1B^2 - A^2B^1)(\mathbf{e}_1 \times \mathbf{e}_2) + (A^1B^3 - A^3B^1)(\mathbf{e}_1 \times \mathbf{e}_3) \\
&+ (A^2B^3 - A^3B^2)(\mathbf{e}_2 \times \mathbf{e}_3) \\
\\
&= (A^1B^2 - A^2B^1)(V\mathbf{e}^3) + (A^1B^3 - A^3B^1)(-V\mathbf{e}_2) \\
&+ (A^2B^3 - A^3B^2)(V\mathbf{e}_1) \\
\\
&= (A^2B^3 - A^3B^2)V\mathbf{e}_1 + (A^3B^1 - A^1B^3)V\mathbf{e}_2 \\
&+ (A^1B^2 - A^2B^1)V\mathbf{e}^3.
\end{aligned}$$

Example 3

Finally, notice that

$$\mathbf{C} = \mathbf{A} \times \mathbf{B} = (A^2B^3 - A^3B^2)V\mathbf{e}_1 + (A^3B^1 - A^1B^3)V\mathbf{e}_2 + (A^1B^2 - A^2B^1)V\mathbf{e}^3$$

$$= \sum_{(i,j,k)} (A^jB^k - A^kB^j)V\mathbf{e}^i,$$

where the sum ranges over all cyclic permutations of $(1, 2, 3)$.

Comparing this formula with $\mathbf{C} = C_i\mathbf{e}^i$, we see that

$$C_i = (A^jB^k - A^kB^j),$$

where (i, j, k) is a cyclic permutation of $(1, 2, 3)$.

Practical question

Your turn!

Express the scalar triple product

$$(\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C}$$

in terms of the **contravariant components** of the vectors $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$, with respect to dual bases. That is

$$\mathbf{A} = A^i \mathbf{e}_i, \qquad \mathbf{B} = B^i \mathbf{e}_i, \qquad \mathbf{C} = C^i \mathbf{e}_i.$$

You may use the result from the previous example:

$$\mathbf{D} = \mathbf{A} \times \mathbf{B} = D_i \mathbf{e}^i = \sum_i (A^j B^k - A^k B^j) V \mathbf{e}^i$$

where

$$V = \mathbf{e}_i \cdot (\mathbf{e}_j \times \mathbf{e}_k)$$

Next Lecture

Next time...

- Chapter 3: Local Coordinate Transform.
  - Arc length and the metric tensor