

TENSOR ANALYSIS

SLIDES WEEK 23 – LECTURE 1

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2025/26

Portfolio

- The **Portfolio Test** will take place on 11 March, 2026.
 - ▶ Duration: 1:15 minutes (+25 minutes if you are entitled to extra time).
 - ▶ Main Cohort: INB2101
 - ▶ PASS or Reasonable adjustments: DCB1105
- The content covers everything we have learned up to the end of this week.

Portfolio overview

Part A — Take-home (25%)

- Released **this Wednesday**.
- Complete at your own pace at home.
- You must **bring Part A** with you to the TCA.

Part B — Time Constrained Assessment (TCA) (75%)

- Takes place on **11 March 2026**

Submission

- Submit **Part A and Part B together** as a **single PDF file**.
- If you cannot submit both together, **e-mail part A** to me.

CHAPTER 3: LOCAL COORDINATE TRANSFORM

Today: Chapter 3–Local Coordinate Transform

1. Preliminaries,
2. Dual bases,
3. Covariant and contravariant components of a vector,
4. The transformation rule,
5. The relationship between covariant and contravariant components, and
6. Arc length and the metric tensor.

Today: Chapter 4–Tensors

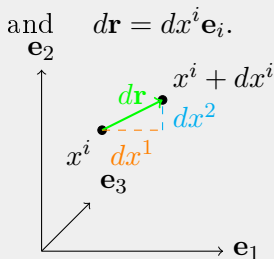
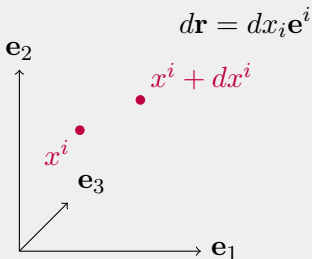
1. Preliminaries,
2. The quotient rule,
3. Symmetric and antisymmetric tensors.

ARC LENGTH AND THE METRIC TENSOR

OUR SET-UP

Consider the coordinate system with basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ and corresponding coordinates x^1, x^2, x^3 .

- Let ds be the arc length between two infinitely close points x^i and $x^i + dx^i$.
- Let the vector $d\mathbf{r}$ join the two points.
- $d\mathbf{r}$ has covariant components dx_i and contravariant components dx^i , that is,



Arc length ds

By definition, we have

$$(ds)^2 = |d\mathbf{r}|^2 = d\mathbf{r} \cdot d\mathbf{r}.$$

Now, because

$$d\mathbf{r} = dx_i \mathbf{e}^i \quad \text{and} \quad d\mathbf{r} = dx^i \mathbf{e}_i,$$

we can write

$$(ds)^2 = \underbrace{\mathbf{e}_i dx^i \cdot \mathbf{e}_k dx^k}_{\text{contravariant}} = g_{ik} dx^i dx^k,$$

or

$$(ds)^2 = \underbrace{\mathbf{e}^i dx_i \cdot \mathbf{e}^k dx_k}_{\text{covariant}} = g^{ik} dx_i dx_k.$$

Arc length ds

We can also write

$$(ds)^2 = \underbrace{\mathbf{e}^i dx_i \cdot \mathbf{e}_k dx^k}_{\text{mix}} = dx_i dx^i$$

because $g_i^i = 1$.

Thus, we end up with the formulas

$$(ds)^2 = g_{ik} dx^i dx^k,$$

$$(ds)^2 = g^{ik} dx_i dx_k,$$

$$(ds)^2 = dx_i dx^i.$$

Arc length and the metric tensor

These equations

$$(ds)^2 = g_{ik} dx^i dx^k,$$

$$(ds)^2 = g^{ik} dx_i dx_k,$$

$$(ds)^2 = dx_i dx^i$$

express the square of the element of arc length in the given coordinate system in terms of g_{ik} (or g^{ik}).

The quantities g_{ik} (or g^{ik}) ‘determine the metric’ of the given space and are known as the **metric tensor**:

- g_{ik} is the covariant metric tensor,
- g^{ik} is the contravariant metric tensor,
- g_i^k is the mixed metric tensor.

Orthogonal Bases

Orthogonal bases are particularly important since the coordinate systems most commonly used in physics and applied mathematics are orthogonal.

We now consider a basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ and its dual basis $\mathbf{e}^1, \mathbf{e}^2, \mathbf{e}^3$ to be both **orthogonal**.

Orthogonal Bases and the metric tensors

For this orthogonal basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$, we have

$$g_{ik} = \mathbf{e}_i \cdot \mathbf{e}_k = g^{ki} = \mathbf{e}^i \cdot \mathbf{e}^k = 0 \quad \text{if } i \neq k.$$

Thus, the general expansion of a vector \mathbf{A}

$$A_i = g_{ik} A^k, \quad \text{and} \quad A^i = g^{ik} A_k$$

reduces to the case $i = k$:

$$\begin{aligned} A_1 &= g_{11} A^1, & A_2 &= g_{22} A^2, & A_3 &= g_{33} A^3 \\ A^1 &= g^{11} A_1, & A^2 &= g^{22} A_2, & A^3 &= g^{33} A_3. \end{aligned}$$

Thus there is a simple relationship between g_{ii} and g^{ii} .

ORTHOGONAL BASES AND THE MATRIC TENSORS - PART 2

Orthogonal Bases and the metric tensors

We have

$$\begin{aligned}A_1 &= g_{11}A^1, & A_2 &= g_{22}A^2, & A_3 &= g_{33}A^3 \\A^1 &= g^{11}A_1, & A^2 &= g^{22}A_2, & A^3 &= g^{33}A_3.\end{aligned}$$

In particular,

$$A_1 = g_{11}A^1 \quad \text{implies} \quad A^1 = \frac{A_1}{g_{11}},$$

but also, we have $A^1 = g^{11}A_1$. Thus,

$$\frac{A_1}{g_{11}} = g^{11}A_1 \quad \text{which implies} \quad g^{11} = \frac{1}{g_{11}}.$$

ORTHOGONAL BASES AND THE METRIC TENSORS - FORMULAS

Orthogonal Bases and the metric tensors

Conclusion: if $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ is an orthogonal basis, then

$$g_{11} = \frac{1}{g^{11}}, \quad g_{22} = \frac{1}{g^{22}}, \quad g_{33} = \frac{1}{g^{33}}.$$

Orthogonal Bases

Going back to our arc length equations

$$(ds)^2 = g_{ik} dx^i dx^k,$$

$$(ds)^2 = g^{ik} dx_i dx_k,$$

$$(ds)^2 = dx_i dx^i,$$

for $i = k$ we can rewrite them as

$$(ds)^2 = \sum_{i=1}^3 (h_i dx^i)^2,$$

where the quantities

$$h_1 = \sqrt{g_{11}}, \quad h_2 = \sqrt{g_{22}}, \quad h_3 = \sqrt{g_{33}}$$

are called the **metric coefficients**.

Example

In a system of **Cartesian coordinates** x_1, x_2, x_3 with orthonormal basis $\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3$, the element of arc length is given by

$$(ds)^2 = g_{jk} dx_j dx_k = \mathbf{i}_j \cdot \mathbf{i}_k dx_j dx_k = \delta_{jk} dx_j dx_k$$

in terms of δ_{jk} .

In other words,

$$(ds)^2 = \sum_{i=1}^3 (h_i dx_i)^2 = (dx_1)^2 + (dx_2)^2 + (dx_3)^2,$$

and the metric coefficients satisfy $h_1 = h_2 = h_3 = 1$.

Finding the basis vectors

We can determine the basis vectors from the position vector \mathbf{r} .

- On the one hand, we have seen that

$$(ds)^2 = |d\mathbf{r}|^2 = d\mathbf{r} \cdot d\mathbf{r} = (\mathbf{e}_i dx^i) \cdot (\mathbf{e}_k dx^k) = (\mathbf{e}_i \cdot \mathbf{e}_k) dx^i dx^k.$$

- On the other hand,

$$\begin{aligned}(ds)^2 &= |d\mathbf{r}|^2 = d\mathbf{r} \cdot d\mathbf{r} = \left(\frac{\partial \mathbf{r}}{\partial x^i} dx^i \right) \cdot \left(\frac{\partial \mathbf{r}}{\partial x^k} dx^k \right) \\ &= \left(\frac{\partial \mathbf{r}}{\partial x^i} \cdot \frac{\partial \mathbf{r}}{\partial x^k} \right) dx^i dx^k.\end{aligned}$$

Therefore the basis vectors are $\mathbf{e}_i = \frac{\partial \mathbf{r}}{\partial x^i}$.

Finding the basis vectors

Since the basis vectors are

$$\mathbf{e}_i = \frac{\partial \mathbf{r}}{\partial x^i},$$

the metric tensor is

$$g_{ij} = \mathbf{e}_i \cdot \mathbf{e}_j = \frac{\partial \mathbf{r}}{\partial x^i} \cdot \frac{\partial \mathbf{r}}{\partial x^j} = \frac{\partial r_\ell}{\partial x^i} \frac{\partial r_\ell}{\partial x^j},$$

where $\mathbf{r} = (x^1, x^2, x^3)$ are the Cartesian coordinates of \mathbf{r} .

SUMMARY: ORTHOGONAL BASES, THE METRIC TENSOR AND ARC LENGTH

Summary

- **Orthogonal** coordinate system (x^1, x^2, x^3) and position vector \mathbf{r} .
- Basis vectors $\mathbf{e}_i = \frac{d\mathbf{r}}{dx^i}$.
- The metric tensor is given by

$$g_{ik} = \mathbf{e}_i \cdot \mathbf{e}_k \neq 0 \text{ if } i = k.$$

- The arc length is then

$$(ds)^2 = g_{ik} dx^i dx^k = \mathbf{e}_i \cdot \mathbf{e}_k dx^i dx^k = \sum_{i=1}^3 (h_i dx^i)^2,$$

where the metric coefficients are $h_i = \sqrt{g_{ii}} = \sqrt{\mathbf{e}_i \cdot \mathbf{e}_i}$.

Example

In **cylindrical coordinates** $(x^1, x^2, x^3) = (r, \theta, z)$, we have the position vector \mathbf{r} is given by

$$\mathbf{r} = r \cos \theta \mathbf{i}_1 + r \sin \theta \mathbf{i}_2 + z \mathbf{i}_3$$

where $\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3$ are the usual Cartesian basis vectors.

The basis vectors are then

$$\mathbf{e}_1 = \frac{\partial \mathbf{r}}{\partial r} = \cos \theta \mathbf{i}_1 + \sin \theta \mathbf{i}_2$$

$$\mathbf{e}_2 = \frac{\partial \mathbf{r}}{\partial \theta} = -r \sin \theta \mathbf{i}_1 + r \cos \theta \mathbf{i}_2$$

$$\mathbf{e}_3 = \frac{\partial \mathbf{r}}{\partial z} = \mathbf{i}_3.$$

EXAMPLE - PART 2

Example

The basis vectors are

$$\mathbf{e}_1 = \cos \theta \mathbf{i}_1 + \sin \theta \mathbf{i}_2, \quad \mathbf{e}_2 = -r \sin \theta \mathbf{i}_1 + r \cos \theta \mathbf{i}_2, \quad \mathbf{e}_3 = \mathbf{i}_3,$$

and the metric tensor is

$$g_{ij} = \mathbf{e}_i \cdot \mathbf{e}_j = 0 \text{ if } i \neq j$$

because the system is **orthogonal**.

We can find the arc length element

$$(ds)^2 = g_{ij} dx^i dx^j = g_{11} dx^1 dx^1 + g_{22} dx^2 dx^2 + g_{33} dx^3 dx^3.$$

Example

The basis vectors are

$$\mathbf{e}_1 = \cos \theta \mathbf{i}_1 + \sin \theta \mathbf{i}_2, \quad \mathbf{e}_2 = -r \sin \theta \mathbf{i}_1 + r \cos \theta \mathbf{i}_2, \quad \mathbf{e}_3 = \mathbf{i}_3.$$

The metric tensor is

$$g_{11} = \mathbf{e}_1 \cdot \mathbf{e}_1 = \cos^2 \theta + \sin^2 \theta = 1,$$

$$g_{22} = \mathbf{e}_2 \cdot \mathbf{e}_2 = r^2 \sin^2 \theta + r^2 \cos^2 \theta = r^2,$$

$$g_{33} = \mathbf{e}_3 \cdot \mathbf{e}_3 = 1.$$

The arc length element is then

$$\begin{aligned} (ds)^2 &= g_{ij} dx^i dx^j = g_{11} dx^1 dx^1 + g_{22} dx^2 dx^2 + g_{33} dx^3 dx^3 \\ &= (dr)^2 + (rd\theta)^2 + (dz)^2. \end{aligned}$$

EXAMPLE - PART 4

Example

We have shown that the arc length element is

$$(ds)^2 = (dr)^2 + (rd\theta)^2 + (dz)^2 = \sum_{i=1}^3 (h_i dx^i)^2.$$

Thus,

$$h_1 = 1, \quad h_2 = r, \quad h_3 = 1.$$

PRACTICAL QUESTION

Your turn!

In spherical coordinates $(x^1, x^2, x^3) = (r, \phi, \theta)$, we have the position vector \mathbf{r} is given by

$$\mathbf{r} = r \sin \phi \cos \theta \mathbf{i}_1 + r \sin \phi \sin \theta \mathbf{i}_2 + r \cos \phi \mathbf{i}_3.$$

Find the basis vectors and the metric coefficients.

TENSORS - PRELIMINARIES

Scalars, Vectors, and Tensors

Today, we will focus on **Cartesian** coordinate systems.

Scalars and vectors are special cases of a more general mathematical object known as a **tensor**.

As we discussed in the first lecture:

1. A scalar is a rank 0 tensor, representing a single value (e.g., temperature at a point).
2. A vector is a rank 1 tensor, representing a quantity with both magnitude and direction (e.g., velocity).
3. A 3×3 matrix is a rank 2 tensor, which can represent more complex relationships (e.g., the moment of inertia in mechanics).
4. And so on.

Components

In **3-dimensional space**, we have:

1. Scalars have one component, e.g., λ
($3^0 = 1$ component).
2. Vectors have three components, e.g., $\mathbf{v} = (v_1, v_2, v_3)$
($3^1 = 3$ components).

In general, a 3D tensor has 3^n components, where n is the rank of the tensor.

Remark: Higher Dimensions

More generally, in m -dimensions, an m -dimensional tensor has m^n components, where n is the rank of the tensor.

FORMAL DEFINITION OF A VECTOR

Formal definition

A **vector** is a quantity satisfying:

$$v'_i = L_{ij}v_j.$$

That is, a vector is a quantity which **transforms** in a certain way under a rotation of coordinates.

Transformation (rotation) matrix L_{ij}

The matrix L_{ij} has the special property $L^T = L^{-1}$. This allows us to show

$$L_{ij}(L_{jk})^T = L_{ik}L_{kj} = \delta_{ik}.$$

Formal definition

We also have shown that

$$x'_i = L_{ij}x_j,$$

which implies

$$L_{ij} = \frac{\partial x'_i}{\partial x_j}.$$

Next steps

In this chapter we want to extend this definition to a more general class of objects, which are called **tensors**.

DEFINITION: TENSOR

Definition

A quantity is a **tensor** if each of the free suffices transform in a certain way under rotation of the coordinates.

For instance,

- **Rank 2:** $T'_{ij} = L_{im}L_{jn}T_{mn}$,
- **Rank 3:** $T'_{ijk} = L_{im}L_{jn}L_{kp}T_{mnp}$,
- **Rank 4:** $T'_{ijkl} = L_{im}L_{jn}L_{kp}L_{lq}T_{mnpq}$, etc.

Note that on the RHS m , n , p and q are repeated dummy indices (summed over), and i , j , k , and l are free indices.

Thus, the equation balances.

DEFINITION: RANK

Definition

The **rank** or **order** of a tensor is the number of free indices.

Example

A 3×3 matrix \mathbf{M} is written as M_{ij} with two free indices i and j which act as row and column counters.

Thus, M_{ij} is a **second-rank** tensor.

Remark

A tensor may have any number of free indices.

For instance, a rank 7 tensor $P_{ijklmnp}$ transforms via:

$$P'_{ijklmnp} = L_{ia}L_{jb}L_{kc}L_{ld}L_{me}L_{nf}L_{pg}P_{abcdefgh}.$$

PRACTICAL QUESTION

Your turn!

Write down the transformation rule for:

1. a tensor of rank four,
2. a tensor of rank six.

EXAMPLE: KRONECKER DELTA

Example: Kronecker Delta

The Kronecker Delta δ_{ij} is a **second-rank** tensor.

To show this, we must show

$$\delta'_{ij} = L_{ik}L_{jm}\delta_{km}.$$

In fact, recall that $L_{ik}L_{jk} = \delta_{ij}$. Thus,

$$L_{ik}L_{jm}\delta_{km} = L_{ik}L_{jk} = \delta_{ij}.$$

Since δ_{ij} is defined in the same in any coordinate system, $\delta_{ij} = \delta'_{ij}$. Thus,

$$L_{ik}L_{jm}\delta_{km} = \delta'_{ij}.$$

Next time...

- Chapter 4: Tensors.
 - ▶ The quotient rule,
 - ▶ Symmetric and antisymmetric tensors.