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A.1.a. $(\underline{a}\times\underline{b})\cdot(\underline{c}\times\underline{d})\underline{a} \; : \; \underline{a},\underline{b},\underline{c},\underline{d}\in\mathbb{R}^3$

$\hookrightarrow a_n[(\underline{a}\times\underline{b})(\underline{c}\times\underline{d})]_i$: by introducing indices/re-arranging

$= a_n(\underline{a}\times\underline{b})_i(\underline{c}\times\underline{d})_i$: by distribution

$= a_n\epsilon_{ijk}a_jb_k\epsilon_{ilm}c_ld_m$: using $(\underline{a}\times\underline{b})_i=\epsilon_{ijk}a_jb_k$

$= a_n[(\epsilon_{ijk}\epsilon_{ilm})a_jb_kc_ld_m]$: by re-arranging

$= a_n[(\epsilon_{jki}\epsilon_{ilm})a_jb_kc_ld_m]$: by cyclic permutation of $\epsilon_{ijk}\to\epsilon_{jki}$

$= a_n[(\delta_{jl}\delta_{km}-\delta_{jm}\delta_{kl})a_jb_kc_ld_m]$: using $\epsilon_{ijk}\epsilon_{klm}=\delta_{il}\delta_{jm}-\delta_{im}\delta_{jl}$

$= a_n(a_jb_kc_jd_k-a_jb_kc_kd_j)$: by distribution

$= a_n[(a_jc_j)(b_kd_k)-(a_jd_j)(b_kc_k)]$: by re-arranging

$= \underline{a}[(\underline{a}\cdot\underline{c})(\underline{b}\cdot\underline{d})-(\underline{a}\cdot\underline{d})(\underline{b}\cdot\underline{c})]$: using $a_ib_i=\underline{a}\cdot\underline{b}$ / converting back to vector

b.i. $\left(\dfrac{\partial T_i}{\partial x^k}\right)' = \dfrac{\partial T_i'}{\partial x'^k}$: by priming

$= \dfrac{\partial}{\partial x'^k}\left(T_i'\right)$: by re-arranging

$= \dfrac{\partial}{\partial x'^k}\left(L_{im}T_m\right)$: using transformation rule

$= \dfrac{\partial}{\partial x'^k}\left(L_{im}\right)T_m + L_{im}\dfrac{\partial T_m}{\partial x'^k}$: by product rule

$= \dfrac{\partial}{\partial x'^k}\left(\dfrac{\partial x^m}{\partial x'^i}\right)T_m + L_{im}\left(\dfrac{\partial T_m}{\partial x^n}\dfrac{\partial x^n}{\partial x'^k}\right)$: using $L_{im}=\dfrac{\partial x^m}{\partial x'^i}$ / chain rule

$= \dfrac{\partial^2 x^m}{\partial x'^k\partial x'^i}T_m + L_{im}\dfrac{\partial T_m}{\partial x^n}L_{kn}$: by combining / using $L_{im}=\dfrac{\partial x^m}{\partial x'^i}$

$= \dfrac{\partial^2 x^m}{\partial x'^k\partial x'^i}T_m + L_{im}L_{kn}\dfrac{\partial T_m}{\partial x^n}$ □ : by re-arranging.

ii. $\dfrac{\partial T_i}{\partial x^k}$ has two free indices, i and k. Hence, it would need to satisfy a rank two transformation rule...

$\hookrightarrow L_{im}L_{kn}\dfrac{\partial T_m}{\partial x^n} = \left(\dfrac{\partial T_i'}{\partial x'^k}\right)'$

$\neq L_{im}L_{kn}\dfrac{\partial T_m}{\partial x^n} + \dfrac{\partial^2 x^m}{\partial x'^k\partial x'^i}T_m$ from part (i)

$\therefore$ not a tensor. □

iii. $A_{ik}' = \left(\dfrac{\partial T_i}{\partial x^k}\right)' - \left(\dfrac{\partial T_k}{\partial x^i}\right)'$

$= \left(L_{im}L_{kn}\dfrac{\partial T_m}{\partial x^n} + \underbrace{\dfrac{\partial^2 x^m}{\partial x'^k\partial x'^i}T_m\right) - \left(\dfrac{\partial^2 x^m}{\partial x'^k\partial x'^i}T_m}_{=0} + \underbrace{L_{im}L_{kn}\dfrac{\partial T_n}{\partial x^m}}_{\text{re-labelling } n\leftrightarrow m \,\because\, \text{dummy indices}}\right)$: from part (i)/ re-arranging

$= L_{im}L_{kn}\left(\dfrac{\partial T_m}{\partial x^n}-\dfrac{\partial T_n}{\partial x^m}\right)$: by simplification

$= L_{im}L_{kn}A_{mn}$: using $A_{ik}=\dfrac{\partial T_i}{\partial x^k}-\dfrac{\partial T_k}{\partial x^i}$

$\hookrightarrow$ The second-rank transformation rule

$\therefore$ A second-rank tensor. □