Tensor Analysis – Practical 11

Solutions

Information:

- These exercises are design to revise the whole module.
- The exercises are **not organised by difficulty**.

11.1 Write the vector equation

$$\mathbf{u} + (\mathbf{a} \cdot \mathbf{b})\mathbf{v} = |\mathbf{a}|^2(\mathbf{b} \cdot \mathbf{v})\mathbf{a}$$

in suffix notation.

Solution:

Step 1 (Free index): We start by adding free indices. We see that this quantity is a vector, so

$$(\mathbf{u} + (\mathbf{a} \cdot \mathbf{b})\mathbf{v})_i = u_i + (\mathbf{a} \cdot \mathbf{b})v_i.$$

Step 2 (Dummy indices): Now, we add the dummy indices:

$$(\mathbf{u} + (\mathbf{a} \cdot \mathbf{b})\mathbf{v})_i = u_i + a_k b_k v_i.$$

We will now write the RHS in suffix notation.

Step 1 (Free index): We start by adding free indices. We see that this quantity is a vector, so

$$(|\mathbf{a}|^2(\mathbf{b} \cdot \mathbf{v})\mathbf{a})_i = |\mathbf{a}|^2(\mathbf{b} \cdot \mathbf{v})a_i.$$

Step 2 (Dummy indices): Now, we add the dummy indices:

$$(|\mathbf{a}|^2(\mathbf{b} \cdot \mathbf{v})\mathbf{a})_i = |\mathbf{a}|^2 b_k v_k a_i.$$

Now, we also know that $|\mathbf{a}| = \sqrt{\mathbf{a} \cdot \mathbf{a}}$, thus

$$|\mathbf{a}|^2 = \mathbf{a} \cdot \mathbf{a} = a_n a_n.$$

It follows that

$$(|\mathbf{a}|^2(\mathbf{b} \cdot \mathbf{v})\mathbf{a})_i = a_n a_n b_k v_k a_i.$$

Consequently, the given expression can be written in suffix notation as

$$u_i + a_k b_k v_i = a_n a_n b_k v_k a_i.$$

11.2 Let f be a scalar field. Using suffix notation, evaluate the following expression

$$\nabla \times (\nabla f).$$

Solution: First, recall that the gradient of a scalar field f is defined by

$$(\nabla f)_k = \frac{\partial f}{\partial x_k}.$$

Thus,

$$\begin{aligned}
(\nabla \times (\nabla f))_i &= \epsilon_{ijk} \nabla_j \left(\frac{\partial f}{\partial x_k} \right) \\
&= \epsilon_{ijk} \frac{\partial}{\partial x_j} \left(\frac{\partial f}{\partial x_k} \right) \\
&= \epsilon_{ijk} \frac{\partial^2 f}{\partial x_j \partial x_k} \\
&= \epsilon_{ijk} \frac{\partial^2 f}{\partial x_k \partial x_j} \\
&= \epsilon_{ikj} \frac{\partial^2 f}{\partial x_j \partial x_k} \qquad (\text{ swapping } j \leftrightarrow k) \\
&= -\epsilon_{ijk} \frac{\partial^2 f}{\partial x_j \partial x_k}.
\end{aligned}$$

In particular,

$$\epsilon_{ijk} \frac{\partial^2 f}{\partial x_j \partial x_k} = -\epsilon_{ijk} \frac{\partial^2 f}{\partial x_j \partial x_k},$$

thus $\epsilon_{ijk} \frac{\partial^2 f}{\partial x_j \partial x_k} = 0$. Thus, we conclude:

$$(\nabla \times (\nabla f))_i = 0$$

11.3 Write the transformation law for the following tensors.

(*1*) A rank 5 tensor in Cartesian coordinates.

(*2*) A rank 5 contravariant tensor in generalised coordinates.

(*3*) The rank 5 mixed tensor $T^{k\ell\cdot\cdot p}_{\cdot\cdot mn\cdot}$ in generalised coordinated.

Solution: **(1)** The transformation rule for a rank 5 tensor $R_{ijk\ell m}$ in Cartesian coordinates is

$$R'_{ijk\ell m} = L_{ia} L_{jb} L_{kc} L_{\ell d} L_{me} R_{abcde}.$$

(2) The transformation rule for a contravariant rank 5 tensor $S^{ijk\ell m}$ in generalised coordinates is

$$S'^{ijk\ell m} = L^{i'}_{a} L^{j'}_{b} L^{k'}_{c} L^{\ell'}_{d} L^{m'}_{e} S^{abcde}.$$

(3) The transformation rule for a rank 5 tensor $T^{k\ell\cdot\cdot p}_{\cdot\cdot mn\cdot}$ in Cartesian coordinates is

$$T'^{k\ell\cdot\cdot p}_{\cdot\cdot mn\cdot} = L^{k'}_{a} L^{\ell'}_{b} L^{c}_{m'} L^{d}_{n'} L^{p'}_{e} T'^{ab\cdot\cdot e}_{\cdot\cdot cd\cdot}.$$

11.4 Given a Cartesian coordinate system K with orthonormal basis $\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3$, consider the coordinate system K' with basis vectors

$$\begin{aligned} \mathbf{e}_1 &= \mathbf{i}_1 + \mathbf{i}_3, \\ \mathbf{e}_2 &= \mathbf{i}_2 - \mathbf{i}_3, \\ \mathbf{e}_3 &= \mathbf{i}_1 - \mathbf{i}_2 + \mathbf{i}_3. \end{aligned}$$

(1) Compute the dual basis vectors $\mathbf{e}^1, \mathbf{e}^2, \mathbf{e}^3$.

(2) Compute the covariant components of the vector $\mathbf{V} = \mathbf{i}_1 - 2\mathbf{i}_3$ with respect to the bases $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ and $\mathbf{e}^1$, $\mathbf{e}^2$, $\mathbf{e}^3$ of part (a).

(3) Compute the contravariant components of the vector $\mathbf{V} = \mathbf{i}_1 - 2\mathbf{i}_3$ with respect to the bases $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ and $\mathbf{e}^1$, $\mathbf{e}^2$, $\mathbf{e}^3$ of part (a).

(4) Find the components of the rotation matrix $L = (L^{j}_{i'})$.

(5) Express the covariant components of the second-order tensor of K with components

$$[P_{ik}] = [P^{ik}] = [P_{i}^{\cdot k}] = [P^{i}_{\cdot k}] = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

in the coordinate system K'.

Solution: **(1)** Using the dual basis formula, we obtain:

$$\mathbf{e}^1 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{e}^2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \quad \mathbf{e}^3 = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}.$$

(2) The covariant components of the vector $\mathbf{V} = \mathbf{i}_1 - 2\mathbf{i}_3$ are given by $V_i = \mathbf{V} \cdot \mathbf{e}_i$. Thus, they are

$$J_1 = -1, \qquad J_2 = 2, \qquad J_3 = -1.$$

(3) The contravariant components of the vector $\mathbf{V} = \mathbf{i}_1 - 2\mathbf{i}_3$ are given by $V^i = \mathbf{V} \cdot \mathbf{e}^i$. Thus, they are

$$J^1 = -2, \qquad J_2 = 3, \qquad J_3 = 3.$$

(4) Let $L = [L^k_{j'}]$ be a matrix. We can either compute each entry of L using the formula $L^k_{j'} = \mathbf{e}_j \cdot \mathbf{i}_k$, or noting that L is the matrix satisfying

$$\begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix} = L \begin{pmatrix} \mathbf{i}_1 \\ \mathbf{i}_2 \\ \mathbf{i}_3 \end{pmatrix}.$$

We conclude

$$L = [L^k_{j'}] = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}.$$

(5) The covariant components of P_{jk} are given by

$$P'_{jk} = L^\ell_{i'} L^m_{k'} P_{\ell m},$$

where $L = [L^p_{q'}]$ is the matrix we computed in the previous item.

Let $P' = [P'_{jk}]$. We have seen in class that

$$\begin{aligned} P' &= [P'_{jk}] = [L^\ell_{i'} L^m_{k'} P_{\ell m}] \\ &= [L^\ell_{i'} P_{\ell m} L^m_{k'}] \\ &= [(L \cdot P)_{im} L^m_{k'}] \\ & LPL^T, \end{aligned}$$

where $P = [P_{jk}]$. Thus, to find all P'_{jk}, we only need to compute LPL^T.

$$\begin{aligned} LPL^T &= \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & -1 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}^T \\ &= \begin{pmatrix} 0 & 1 & 1 \\ -1 & 0 & -2 \\ 1 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -2 \\ -3 & 2 & 1 \\ 1 & 1 & 0 \end{pmatrix}. \end{aligned}$$

11.5 In this question, consider the two-dimensional orthogonal coordinate system $(x^1, x^2) = (x, \theta)$ with position r given by

$$\mathbf{r} = e^x \sin\theta \mathbf{i}_1 + e^x \cos\theta \mathbf{i}_2,$$

where $\mathbf{i}_1, \mathbf{i}_2$ are the usual 2 dimensional Cartesian basis vectors.

(1) Compute the basis vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ of this coordinate system.

(2) Compute the metric tensor g_{ik} of this coordinate system.

(3) Describe the arc length element in terms of the metric coefficients.

(4) Determine the Christoffel symbols of the first kind for the given coordinate system.

(5) Determine the Christoffel symbols of the second kind for the given coordinate system.

(6) Compute the following components of the Riemann-Christoffel tensor

$$R^1_{111}, \qquad R^2_{121}, \qquad R^1_{222}.$$

Solution: **(1)** The basis vectors are given by the formula

$$\mathbf{e}_i = \frac{\partial \mathbf{r}}{\partial x^i},$$

thus

$$\begin{aligned}\mathbf{e}_1 &= \frac{\partial \mathbf{r}}{\partial x^1} = \frac{\partial \mathbf{r}}{\partial x} = e^x \sin\theta \mathbf{i}_1 + e^x \cos\theta \mathbf{i}_2 \\ \mathbf{e}_2 &= \frac{\partial \mathbf{r}}{\partial x^2} = \frac{\partial \mathbf{r}}{\partial \theta} = e^x \cos\theta \mathbf{i}_1 - e^x \sin\theta \mathbf{i}_2.\end{aligned}$$

(2) The metric tensor is defined by

$$g_{ij} = \mathbf{e}_i \cdot \mathbf{e}_j.$$

Thus,

$$\begin{aligned} g_{11} &= \mathbf{e}_1 \cdot \mathbf{e}_1 = \begin{pmatrix} e^x \sin\theta \\ e^x \cos\theta \end{pmatrix} \cdot \begin{pmatrix} e^x \sin\theta \\ e^x \cos\theta \end{pmatrix} = e^{2x}\left(\sin^2\theta + \cos^2\theta\right) = e^{2x}, \\ g_{22} &= \mathbf{e}_2 \cdot \mathbf{e}_2 = \begin{pmatrix} e^x \cos\theta \\ -e^x \sin\theta \end{pmatrix} \cdot \begin{pmatrix} e^x \cos\theta \\ -e^x \sin\theta \end{pmatrix} = e^{2x}\left(\cos^2\theta + \sin^2\theta\right) = e^{2x}, \\ g_{12} &= g_{21} = \mathbf{e}_1 \cdot \mathbf{e}_2 = \begin{pmatrix} e^x \sin\theta \\ e^x \cos\theta \end{pmatrix} \cdot \begin{pmatrix} e^x \cos\theta \\ -e^x \sin\theta \end{pmatrix} = 0. \end{aligned}$$

We conclude

$$G = [g_{ik}] = \begin{pmatrix} e^{2x} & 0 \\ 0 & e^{2x} \end{pmatrix}.$$

(3) Since the system is orthogonal, the arc length element is given by

$$\begin{aligned}(ds)^2 &= g_{11} dx^1 dx^1 + g_{22} dx^2 dx^2 \\ &= (e^x dx)^2 + (e^x d\theta)^2 \\ &= (h_1 dx)^2 + (h_2 d\theta)^2\end{aligned}$$

The metric coefficients are

$$h_1 = e^{\rho}, \qquad h_2 = e^{\rho}\cos\theta, \qquad h_3 = e^{\rho}.$$

(4) The Christoffel symbols of first kind are given by

$$\Gamma_{ijk} = \frac{1}{2}\left(\frac{\partial g_{ij}}{\partial x^k} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^i}\right).$$

The only partial derivatives $\frac{\partial g_{mn}}{\partial x^\ell}$ that are non-zero are

$$\frac{\partial g_{11}}{\partial x^1} = \frac{\partial e^{2x}}{\partial x} = 2e^{2x}, \qquad \frac{\partial g_{22}}{\partial x^1} = \frac{\partial e^{2x}}{\partial x} = 2e^{2x}.$$

Thus, the only non-zero Christoffel symbols of first kind are

$$\Gamma_{111}, \qquad \Gamma_{122}, \qquad \Gamma_{212}, \qquad \Gamma_{221}.$$

We have

$$\Gamma_{111} = \frac{1}{2}\left(\frac{\partial g_{11}}{\partial x^1} + \frac{\partial g_{1q}}{\partial x^1} - \frac{\partial g_{11}}{\partial x^1}\right) = \frac{1}{2}\frac{\partial g_{11}}{\partial x} = e^{2x},$$
$$\Gamma_{122} = \frac{1}{2}\left(\frac{\partial g_{12}}{\partial x^2} + \frac{\partial g_{12}}{\partial x^2} - \frac{\partial g_{22}}{\partial x^1}\right) = -\frac{\partial g_{22}}{\partial x} = -e^{2x}.$$

By Ricci's Theorem $\Gamma_{212} = -\Gamma_{122} = e^{2x}$. By the symmetries of the Christoffel symbols

$$\Gamma_{221} = \Gamma_{212} = 2e^{2x}.$$

(5) To find the Christoffel symbols of second kind, we use the formula

$$\Gamma^i_{jk} = g^{i\ell}\Gamma_{\ell jk}.$$

Thus, we need to compute the g^{ik}. Since

$$[g_{ik}] = \begin{pmatrix} e^{2x} & 0 \\ 0 & e^{2x} \end{pmatrix},$$

we get

$$[g^{ik}] = \begin{pmatrix} e^{-2x} & 0 \\ 0 & e^{-2x} \end{pmatrix}.$$

Note that $\Gamma^i_{jk} = g^{i\ell}\Gamma_{\ell jk}$ is zero if Γ_{ijk} is zero. Thus, the only non-zero Christoffel symbols of second kind are

$$\Gamma^1_{11} = g^{1\ell}\Gamma_{\ell 11} = e^{-2x}(e^{2x}) = 1,$$
$$\Gamma^1_{22} = g^{1\ell}\Gamma_{\ell 22} = e^{-2x}(-e^{2x}) = -1.$$
$$\Gamma^2_{12} = \Gamma^2_{21} = g^{2\ell}\Gamma_{\ell 21} = e^{-2x}(e^{2x}) = 1.$$

(6) The Riemann-Christoffel symbols is given by

$$R^r_{ijk} = \left[\frac{\partial \Gamma^r{}_{ki}}{\partial x^j} - \frac{\partial \Gamma^r{}_{ij}}{\partial x^k} + \Gamma^p{}_{ik}\Gamma^r{}_{pj} - \Gamma^p{}_{ij}\Gamma^r{}_{pk} \right].$$

Since all Christoffel symbols of second kind are constants, this formula simplifies to

$$\begin{aligned} R^r_{ijk} &= \Gamma^p{}_{ik}\Gamma^r{}_{pj} - \Gamma^p{}_{ij}\Gamma^r{}_{pk} \\ &= \Gamma^1{}_{ik}\Gamma^r{}_{1j} + \Gamma^2{}_{ik}\Gamma^r{}_{2j} - \Gamma^1{}_{ij}\Gamma^r{}_{1k} - \Gamma^2{}_{ij}\Gamma^r{}_{2k}. \end{aligned}$$

So

$$\begin{aligned} R^1_{111} &= \Gamma^1{}_{11}\Gamma^1{}_{11} + \underbrace{\Gamma^2{}_{11}\Gamma^1{}_{21}}_{=0} - \Gamma^1{}_{11}\Gamma^1{}_{11} - \underbrace{\Gamma^2{}_{11}\Gamma^1{}_{21}}_{=0} = 0, \\ R^2_{121} &= \Gamma^1{}_{11}\Gamma^2{}_{12} + \underbrace{\Gamma^2{}_{11}\Gamma^2{}_{22}}_{=0} - \underbrace{\Gamma^1{}_{12}\Gamma^2{}_{11}}_{=0} - \Gamma^2{}_{12}\Gamma^2{}_{21} = 1 - 1 = 0, \\ R^1_{222} &= \underbrace{\Gamma^1{}_{22}\Gamma^1{}_{12}}_{=0} + \underbrace{\Gamma^2{}_{22}\Gamma^1{}_{22}}_{=0} - \underbrace{\Gamma^1{}_{22}\Gamma^1{}_{12}}_{=0} - \underbrace{\Gamma^2{}_{22}\Gamma^1{}_{22}}_{=0} = 0. \end{aligned}$$
