Tensor Analysis – Practical 11

Information:

- These exercises are design to revise the whole module.
- The exercises are **not organised by difficulty**.

11.1 Write the vector equation

$$\mathbf{u} + (\mathbf{a} \cdot \mathbf{b})\mathbf{v} = |\mathbf{a}|^2(\mathbf{b} \cdot \mathbf{v})\mathbf{a}$$

in suffix notation.

11.2 Let f be a scalar field. Using suffix notation, evaluate the following expression

$$\nabla \times (\nabla f).$$

11.3 Write the transformation law for the following tensors.

(1) A rank 5 tensor in Cartesian coordinates.

(2) A rank 5 contravariant tensor in generalised coordinates.

(3) The rank 5 mixed tensor $T^{k\ell\cdot\cdot p}_{\cdot\cdot mn\cdot}$ in generalised coordinated.

11.4 Given a Cartesian coordinate system K with orthonormal basis $\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3$, consider the coordinate system K' with basis vectors

$$\begin{aligned}\mathbf{e}_1 &= \mathbf{i}_1 + \mathbf{i}_3,\\ \mathbf{e}_2 &= \mathbf{i}_2 - \mathbf{i}_3,\\ \mathbf{e}_3 &= \mathbf{i}_1 - \mathbf{i}_2 + \mathbf{i}_3.\end{aligned}$$

(1) Compute the dual basis vectors $\mathbf{e}^1, \mathbf{e}^2, \mathbf{e}^3$.

(2) Compute the covariant components of the vector $\mathbf{V} = \mathbf{i}_1 - 2\mathbf{i}_3$ with respect to the bases $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ and $\mathbf{e}^1$, $\mathbf{e}^2$, $\mathbf{e}^3$ of part (a).

(3) Compute the contravariant components of the vector $\mathbf{V} = \mathbf{i}_1 - 2\mathbf{i}_3$ with respect to the bases $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ and $\mathbf{e}^1$, $\mathbf{e}^2$, $\mathbf{e}^3$ of part (a).

(4) Find the components of the rotation matrix $L = (L^j_{i'})$.

(5) Express the covariant components of the second-order tensor of K with components

$$[P_{ik}] = [P^{ik}] = [P_i^{\cdot k}] = [P^i_{\cdot k}] = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

in the coordinate system K'.

11.5 In this question, consider the two-dimensional orthogonal coordinate system $(x^1, x^2) = (x, \theta)$ with position r given by

$$\mathbf{r} = e^x \sin\theta \mathbf{i}_1 + e^x \cos\theta \mathbf{i}_2,$$

where $\mathbf{i}_1, \mathbf{i}_2$ are the usual 2 dimensional Cartesian basis vectors.

(*1*) Compute the basis vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ of this coordinate system.

(*2*) Compute the metric tensor g_{ik} of this coordinate system.

(*3*) Describe the arc length element in terms of the metric coefficients.

(*4*) Determine the Christoffel symbols of the first kind for the given coordinate system.

(*5*) Determine the Christoffel symbols of the second kind for the given coordinate system.

(*6*) Compute the following components of the Riemann-Christoffel tensor

$$R^1_{111}, \qquad R^2_{121}, \qquad R^1_{222}.$$