Tensor Analysis – Practical 2

Solutions

Information:

- Please make sure to complete **all** exercises **before** the next lecture.
- The exercises marked with [**See lecture**] were solved in class.
- The exercises are **not organised by difficulty**.

2.1 Which of the following combinations of vector differential operators are valid?

(1) curl curl, curl grad, div grad.

(2) div grad, div curl, div div.

(3) grad div, curl grad, curl curl, grad grad, div curl.

(4) div grad, div curl, curl grad, div curl.

Solution: (1) and (4).

2.2 Translate the suffix notation equation

$$\delta_{ij}c_j + \epsilon_{kji}a_k b_j = d_\ell e_m c_i b_\ell c_m$$

into ordinary vector notation.

Solution: We know from the lectures that

$$\epsilon_{kji}a_k b_j = (\mathbf{a} \times \mathbf{b})_i.$$

We also have seen that $\delta_{ij}c_j = c_i$. Thus, the LHS is

$$\delta_{ij}c_j + \epsilon_{kji}a_k b_j = \mathbf{c} + (\mathbf{a} \times \mathbf{b}).$$

We can rearrange the terms in $d_\ell e_m c_i b_\ell c_m$ to have repeated indices together, obtaining

$$d_\ell e_m c_i b_\ell c_m = b_\ell d_\ell c_m e_m c_i = (\mathbf{b} \cdot \mathbf{d})(\mathbf{c} \cdot \mathbf{e})\mathbf{c}.$$

It follows that the required vector equation is

$$\mathbf{c} + (\mathbf{a} \times \mathbf{b}) = (\mathbf{b} \cdot \mathbf{d})(\mathbf{c} \cdot \mathbf{e})\mathbf{c}.$$

2.3 [**Question from the Final Exam 23-24**] Let $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$, and $\mathbf{z}$ be vectors in $\mathbb{R}^3$. Using suffix notation, find an expression involving no cross products for

$$(\mathbf{u} \times (\mathbf{v} \times \mathbf{w})) \cdot \mathbf{z}.$$

Write your final answer in **vector notation**. Provide all steps of your workings.

Solution: Using suffix notation, we get

$$\begin{aligned}(\mathbf{u} \times (\mathbf{v} \times \mathbf{w})) \cdot \mathbf{z} &= (\mathbf{u} \times (\mathbf{v} \times \mathbf{w}))_i z_i \\ &= \epsilon_{ijk} u_j (\mathbf{v} \times \mathbf{w})_k z_i \\ &= \epsilon_{ijk} u_j (\epsilon_{k\ell m} v_\ell w_m) z_i \\ &= (\delta_{i\ell}\delta_{jm} - \delta_{im}\delta_{j\ell}) u_j v_\ell w_m z_i \\ &= u_j v_i w_j z_i - u_j v_j w_i z_i \\ &= (\mathbf{u} \cdot \mathbf{w})(\mathbf{v} \cdot \mathbf{z}) - (\mathbf{u} \cdot \mathbf{v})(\mathbf{w} \cdot \mathbf{z}).\end{aligned}$$

2.4 Here, we will compute the gradient of a dot product. In other words, we will take steps to find an expression for $\nabla(\mathbf{u} \cdot \mathbf{v})$.

(*1*) Show that

$$[\mathbf{u} \times (\nabla \times \mathbf{v})]_i = u_j \frac{\partial v_j}{\partial x_i} - u_j \frac{\partial v_i}{\partial x_j}.$$

(*2*) Use item (1) to show

$$[\mathbf{u} \times (\nabla \times \mathbf{v}) + \mathbf{v} \times (\nabla \times \mathbf{u})]_i = u_j \frac{\partial v_j}{\partial x_i} - u_j \frac{\partial v_i}{\partial x_j} + v_j \frac{\partial u_j}{\partial x_i} - v_j \frac{\partial u_i}{\partial x_j}.$$

(*3*) Conclude that

$$\nabla(\mathbf{u} \cdot \mathbf{v}) = \mathbf{u} \times (\nabla \times \mathbf{v}) + \mathbf{v} \times (\nabla \times \mathbf{u}) + (\mathbf{u} \cdot \nabla)\mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{u}.$$

Solution: **(1)** Using the definition of cross product, we get

$$\begin{aligned}[\mathbf{u} \times (\nabla \times \mathbf{v})]_i &= \epsilon_{ijk} u_j (\nabla \times \mathbf{v})_k \\ &= \epsilon_{ijk} u_j \epsilon_{k\ell m} \frac{\partial v_m}{\partial x_\ell} \\ &= (\delta_{i\ell}\delta_{jm} - \delta_{im}\delta_{j\ell}) u_j \frac{\partial v_m}{\partial x_\ell} \\ &= u_j \frac{\partial v_j}{\partial x_i} - u_j \frac{\partial v_i}{\partial x_j}.\end{aligned}$$

(2) We have shown

$$[\mathbf{u} \times (\nabla \times \mathbf{v})]_i = u_j \frac{\partial v_j}{\partial x_i} - u_j \frac{\partial v_i}{\partial x_j}.$$

Similarly,

$$[\mathbf{v} \times (\nabla \times \mathbf{u})]_i = v_j \frac{\partial u_j}{\partial x_i} - v_j \frac{\partial u_i}{\partial x_j}.$$

Adding everything together, we get

$$[\mathbf{u} \times (\nabla \times \mathbf{v}) + \mathbf{v} \times (\nabla \times \mathbf{u})]_i = u_j \frac{\partial v_j}{\partial x_i} - u_j \frac{\partial v_i}{\partial x_j} + v_j \frac{\partial u_j}{\partial x_i} - v_j \frac{\partial u_i}{\partial x_j}$$

(3) In vector notation, we know that

$$u_j \frac{\partial v_i}{\partial x_j} = [(\mathbf{u} \cdot \nabla)\mathbf{v}]_i \quad \text{and } v_j \frac{\partial u_i}{\partial x_j} = [(\mathbf{v} \cdot \nabla)\mathbf{u}]_i.$$

where

$$\mathbf{u} \cdot \nabla = u_j \frac{\partial}{\partial x_j}.$$

Substituting this in the expression we got in (2), we get

$$[\mathbf{u} \times (\nabla \times \mathbf{v}) + \mathbf{v} \times (\nabla \times \mathbf{u})]_i = u_j \frac{\partial v_j}{\partial x_i} - [(\mathbf{u} \cdot \nabla)\mathbf{v}]_i + v_j \frac{\partial u_j}{\partial x_i} - [(\mathbf{v} \cdot \nabla)\mathbf{u}]_i.$$

We then just have to write the remaning two terms in vector notation. We have

$$u_j \frac{\partial v_j}{\partial x_i} + v_j \frac{\partial u_j}{\partial x_i} = [\nabla(\mathbf{u} \cdot \mathbf{v})]_i.$$

So that

$$[\mathbf{u} \times (\nabla \times \mathbf{v}) + \mathbf{v} \times (\nabla \times \mathbf{u})]_i = [\nabla(\mathbf{u} \cdot \mathbf{v}) - (\mathbf{u} \cdot \nabla)\mathbf{v} - (\mathbf{v} \cdot \nabla)\mathbf{u}]_i\,.$$

We then isolate the expression for $\nabla(\mathbf{u} \cdot \mathbf{v})$ to get

$$\nabla(\mathbf{u} \cdot \mathbf{v}) = \mathbf{u} \times (\nabla \times \mathbf{v}) + \mathbf{v} \times (\nabla \times \mathbf{u}) + (\mathbf{u} \cdot \nabla)\mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{u}.$$

2.5 Recall the relation

$$\epsilon_{ijk}\epsilon_{k\ell m} = \delta_{i\ell}\delta_{jm} - \delta_{im}\delta_{j\ell}.$$

Check that the left-hand side is equal to the right-hand side in the following cases.

(1) $i = 1, j = 2, k = 3, \ell = 1, m = 2$,

(2) $i = 2, j = 1, k = 3, \ell = 2, m = 1$.

Solution: (1) $\epsilon_{123}\epsilon_{312} = 1 \cdot 1 = 1$ and $\delta_{11}\delta_{22} - \delta_{12}\delta_{21} = 1 \cdot 1 - 0 \cdot 0 = 1$. (2) Similar.

2.6 Simplify the following suffix notation expressions.

- *(1)* $\delta_{ij}\epsilon_{ijk}$; (Note that this is a vector.)
- *(2)* $\epsilon_{ijk}\epsilon_{i\ell m}$;
- *(3)* $\epsilon_{ijk}\epsilon_{ijm}$;
- *(4)* $\epsilon_{ijk}\epsilon_{ijk}$.

Solution: **(1)** As $\delta_{ij} = 0$ when $i \neq j$, we obtain

$$\delta_{ij}\epsilon_{ijk} = \epsilon_{iik},$$

which is $(\mathbf{0})_k$ (the zero vector) by the definition of the alternating tensor.

(2) Using the relations $\epsilon_{ijk} = \epsilon_{jki}$ and $\epsilon_{abc}\epsilon_{cde} = \delta_{ad}\delta_{be} - \delta_{ae}\delta_{bd}$, we get

$$\begin{aligned}\epsilon_{ijk}\epsilon_{i\ell m} &= \epsilon_{jki}\epsilon_{i\ell m} \\ &= \delta_{j\ell}\delta_{km} - \delta_{jm}\delta_{k\ell}.\end{aligned}$$

(3) Replacing ℓ with j in the above, we obtain

$$\begin{aligned}\epsilon_{ijk}\epsilon_{ijm} &= \epsilon_{jki}\epsilon_{ijm} \\ &= \delta_{jj}\delta_{km} - \delta_{jm}\delta_{kj} \\ &= 3\delta_{km} - \delta_{km} \\ &= 2\delta_{km}\end{aligned}$$

where the second last equality arises from the fact that $\delta_{ab}\delta_{bc} = \delta_{ac}$.

(4) Replacing m with k in (c) gives

$$\epsilon_{ijk}\epsilon_{ijk} = 2\delta_{kk} = 6.$$

2.7 Use the formula

(1) $$\epsilon_{pqr}|M| = \epsilon_{ijk}M_{pi}M_{qj}M_{rk} \quad \text{(see Lecture slides)}$$

to show that

- *(1)* $6|M| = \epsilon_{pqr}\epsilon_{ijk}M_{pi}M_{qj}M_{rk}$,
- *(2)* $|M^T| = |M|$,
- *(3)* $|MN| = |M||N|$.

Solution: **(1)** Upon multiplying formula (1) by ϵ_{pqr} on the left to both sides, we obtain

$$\epsilon_{pqr}\epsilon_{pqr}|M| = \epsilon_{pqr}\epsilon_{ijk}M_{pi}M_{qj}M_{rk}$$

which is equivalent to the following because of the equality $\epsilon_{pqr}\epsilon_{pqr} = 6$ that we have shown in (2.6)(4)

$$6|M| = \epsilon_{pqr}\epsilon_{ijk}M_{pi}M_{qj}M_{rk}.$$

(2) From (a), we have

$$|M| = \frac{1}{6}\epsilon_{pqr}\epsilon_{ijk}M_{pi}M_{qj}M_{rk}.$$

Now

$$|M^T| = \frac{1}{6}\epsilon_{pqr}\epsilon_{ijk}M_{ip}M_{jq}M_{kr},$$

which is equal to

$$\frac{1}{6}\epsilon_{ijk}\epsilon_{pqr}M_{pi}M_{qj}M_{rk} = |M|$$

upon relabeling $i \leftrightarrow p$, $j \leftrightarrow q$ and $k \leftrightarrow r$.

(3) Recalling that, for a matrix $\mathbf{A}$, we have $\epsilon_{pqr}|\mathbf{A}| = \epsilon_{ijk}A_{pi}A_{qj}A_{rk}$, we can write

$$\epsilon_{pqr}|M||N| = \epsilon_{ijk}M_{pi}M_{qj}M_{rk} \cdot |N| = M_{pi}M_{qj}M_{rk} \cdot (\epsilon_{ijk}|N|).$$

Because

$$\epsilon_{ijk}|N| = \epsilon_{stu}N_{is}N_{jt}N_{ku},$$

we get

$$\begin{aligned}\epsilon_{pqr}|M||N| &= M_{pi}M_{qj}M_{rk} \cdot \epsilon_{stu}N_{is}N_{jt}N_{ku} \\ &= \epsilon_{stu}M_{pi}N_{is}M_{qj}N_{jt}M_{rk}N_{ku} \\ &= \epsilon_{stu}(MN)_{ps}(MN)_{qt}(MN)_{ru} \\ &= \epsilon_{pqr}|MN|.\end{aligned}$$

Hence $|MN| = |M||N|$, as required.

2.8 Show that $\nabla f(r) = f'(r)\mathbf{r}/r$, where $\mathbf{r}$ is the position vector $\mathbf{r} = (x_1, x_2, x_3)$ and $r = |\mathbf{r}|$. [Hints: First recall that $f'(r) = \frac{\partial f}{\partial r}$. Then, at some point you should have the expression $\frac{\partial r}{\partial x_i}$ to deal with. Here it is helpful to first separately write out an equation expressing r in terms of x_1, x_2, x_3, and to then write this equation in suffix notation.]

Solution: In suffix notation, we have

$$\begin{aligned}
[\nabla f(r)]_i &= \frac{\partial f(r)}{\partial x_i} \\
&= \frac{\partial f(r)}{\partial r}\frac{\partial r}{\partial x_i} \\
&= f'(r)\frac{\partial}{\partial x_i}(x_j x_j)^{1/2} \qquad (\text{using } r = (x_j x_j)^{1/2}) \\
&= f'(r)\left(\frac{1}{2(x_j x_j)^{1/2}}\left[2x_j\frac{\partial x_j}{\partial x_i}\right]\right) \\
&= f'(r)\left(\frac{1}{r}\left[x_j\frac{\partial x_j}{\partial x_i}\right]\right) \\
&= f'(r)\frac{x_j\delta_{ij}}{r} \qquad\qquad \left(\text{as } \frac{\partial x_j}{\partial x_i} = \delta_{ij}\right) \\
&= f'(r)\frac{x_i}{r} \qquad\qquad (\text{as } \delta_{ij}x_j = x_i) \\
&= [f'(r)\mathbf{r}/r]_i.
\end{aligned}$$

Warning: Writing $r(= \sqrt{x_1^2 + x_2^2 + x_3^3}) = \sqrt{x_j^2}$ is wrong, because as a suffix notation expression, $\sqrt{x_j^2}$ is just x_j. You need to explicitly write $\sqrt{x_j x_j}$. Only when j is visibly repeated, is then summation implied.
