Tensor Analysis – Practical 2

Information:

- Please make sure to complete **all** exercises **before** the next lecture.
- The exercises marked with [**See lecture**] were solved in class.
- The exercises are **not organised by difficulty**.

2.1 Which of the following combinations of vector differential operators are valid?

(*1*) curl curl, curl grad, div grad.

(*2*) div grad, div curl, div div.

(*3*) grad div, curl grad, curl curl, grad grad, div curl.

(*4*) div grad, div curl, curl grad, div curl.

2.2 Translate the suffix notation equation

$$\delta_{ij}c_j + \epsilon_{kji}a_k b_j = d_\ell e_m c_i b_\ell c_m$$

into ordinary vector notation.

2.3 [**Question from the Final Exam 23-24**] Let $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$, and $\mathbf{z}$ be vectors in $\mathbb{R}^3$. Using suffix notation, find an expression involving no cross products for

$$(\mathbf{u} \times (\mathbf{v} \times \mathbf{w})) \cdot \mathbf{z}.$$

Write your final answer in **vector notation**. Provide all steps of your workings.

2.4 Here, we will compute the gradient of a dot product. In other words, we will take steps to find an expression for $\nabla(\mathbf{u} \cdot \mathbf{v})$.

(*1*) Show that

$$[\mathbf{u} \times (\nabla \times \mathbf{v})]_i = u_j \frac{\partial v_j}{\partial x_i} - u_j \frac{\partial v_i}{\partial x_j}.$$

(*2*) Use item (1) to show

$$[\mathbf{u} \times (\nabla \times \mathbf{v}) + \mathbf{v} \times (\nabla \times \mathbf{u})]_i = u_j \frac{\partial v_j}{\partial x_i} - u_j \frac{\partial v_i}{\partial x_j} + v_j \frac{\partial u_j}{\partial x_i} - v_j \frac{\partial u_i}{\partial x_j}.$$

(*3*) Conclude that

$$\nabla(\mathbf{u} \cdot \mathbf{v}) = \mathbf{u} \times (\nabla \times \mathbf{v}) + \mathbf{v} \times (\nabla \times \mathbf{u}) + (\mathbf{u} \cdot \nabla)\mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{u}.$$

2.5 Recall the relation

$$\epsilon_{ijk}\epsilon_{k\ell m} = \delta_{i\ell}\delta_{jm} - \delta_{im}\delta_{j\ell}.$$

Check that the left-hand side is equal to the right-hand side in the following cases.

(1) $i = 1, j = 2, k = 3, \ell = 1, m = 2,$
(2) $i = 2, j = 1, k = 3, \ell = 2, m = 1.$

2.6 Simplify the following suffix notation expressions.

(1) $\delta_{ij}\epsilon_{ijk}$; (Note that this is a vector.)
(2) $\epsilon_{ijk}\epsilon_{i\ell m}$;
(3) $\epsilon_{ijk}\epsilon_{ijm}$;
(4) $\epsilon_{ijk}\epsilon_{ijk}$.

2.7 Use the formula

(1) $$\epsilon_{pqr}|M| = \epsilon_{ijk}M_{pi}M_{qj}M_{rk} \quad \text{(see Lecture slides)}$$

to show that

(1) $6|M| = \epsilon_{pqr}\epsilon_{ijk}M_{pi}M_{qj}M_{rk},$
(2) $|M^T| = |M|,$
(3) $|MN| = |M||N|.$

2.8 Show that $\nabla f(r) = f'(r)\mathbf{r}/r$, where $\mathbf{r}$ is the position vector $\mathbf{r} = (x_1, x_2, x_3)$ and $r = |\mathbf{r}|$. [Hints: First recall that $f'(r) = \frac{\partial f}{\partial r}$. Then, at some point you should have the expression $\frac{\partial r}{\partial x_i}$ to deal with. Here it is helpful to first separately write out an equation expressing r in terms of x_1, x_2, x_3, and to then write this equation in suffix notation.]