Tensor Analysis – Practical 3

Solutions

Information:

- Please make sure to complete **all** exercises **before** the next lecture.
- The exercises marked with **[See lecture]** were solved in class.
- The exercises are **not organised by difficulty**.

3.1 **[See lecture]** Use the **transformation law** to show that ∇ is a vector.

Solution: We must show that

$$\nabla'_i = L_{ij}\nabla_j.$$

Note that

$$\begin{aligned}
[\nabla']_i &= \frac{\partial}{\partial x'_i} \\
&= \frac{\partial x_j}{\partial x'_i}\frac{\partial}{\partial x_j} && \text{(Chain rule)} \\
&= L_{ij}\frac{\partial}{\partial x_j}, && \left(\frac{\partial x_j}{\partial x'_i} = L_{ij}\right).
\end{aligned}$$

As $[\nabla']_i = L_{ij}\frac{\partial}{\partial x_j} = L_{ij}[\nabla]_j$, we have our result.

3.2 **[See lecture]** Let f be a scalar field. Show that $\nabla \cdot (\nabla f)$ is is a scalar using the **transformation law**.

Solution: We must show that

$$(\nabla \cdot (\nabla f))' = \nabla \cdot (\nabla f).$$

Let us first write $(\nabla \cdot (\nabla f))'$ in suffix notation.

$$\begin{aligned}
(\nabla \cdot (\nabla f))' &= \nabla' \cdot (\nabla' f') \\
&= \nabla'_i (\nabla' f')_i \\
&= \nabla'_i \nabla'_i f'.
\end{aligned}$$

Now, since f is a scalar field, its transformation law is $f' = f$. Since ∇ is a vector (see previous exercise) its transformation law is

$$\nabla'_i = L_{ij}\nabla_j.$$

Therefore

$$
\begin{aligned}
(\nabla \cdot (\nabla f))' &= \nabla'_i \nabla'_i f' \\
&= (L_{ij}\nabla_j)(L_{ik}\nabla_k) f \\
&= L_{ij} L_{ik} \nabla_j \nabla_k f \\
&= \delta_{jk} \nabla_j \nabla_k f \\
&= \nabla_k \nabla_k f \\
&= \nabla_k (\nabla f)_k \\
&= \nabla \cdot (\nabla f),
\end{aligned}
$$

which is the transformation law of a scalar quantity.

3.3 Use the **transformation law** and the fact that ∇ and $\mathbf{w}$ are vectors to show that

$$\mathbf{w} \cdot \nabla$$

is a scalar.

Solution: We must show that

$$(\mathbf{w} \cdot \nabla)' = \mathbf{w} \cdot \nabla.$$

We know that $\mathbf{w}$ and ∇ are vectors, thus $w'_i = L_{ij} w_j$ and $\nabla'_i = \frac{\partial}{\partial x'_i} = L_{ik} \frac{\partial}{\partial x_k} = L_{ik} \nabla_k$.

Thus, we get

$$
\begin{aligned}
(\mathbf{w} \cdot \nabla)' &= w'_i \frac{\partial}{\partial x'_i} \\
&= L_{ij} w_j L_{ik} \frac{\partial}{\partial x_k} \\
&= L_{ij} L_{ik} w_j \frac{\partial}{\partial x_k} \\
&= \delta_{jk} w_j \frac{\partial}{\partial x_k} \qquad\qquad (L_{ij} L_{ik} = \delta_{jk}) \\
&= w_j \frac{\partial}{\partial x_j} \\
&= \mathbf{w} \cdot \nabla.
\end{aligned}
$$

3.4 Consider $L_{ij} = \frac{\partial x'_i}{\partial x_j}$. Show that $\frac{\partial L_{ij}}{\partial x'_i} = 0$.

Solution: We have

$$\begin{aligned}\frac{\partial L_{ij}}{\partial x'_i} &= \frac{\partial^2 x'_i}{\partial x'_i \partial x_j} \\ &= \frac{\partial^2 x'_i}{\partial x_j \partial x'_i} \quad \text{(as the order of the differential does not matter)} \\ &= \frac{\partial}{\partial x_j} \frac{\partial x'_i}{\partial x'_i} \\ &= \frac{\partial}{\partial x_j} \delta_{ii} \\ &= \frac{\partial}{\partial x_j}(3) \\ &= 0.\end{aligned}$$

Warning: Note that in the expression $\frac{\partial x'_i}{\partial x'_i}$, the index i is repeated, so this expression is actually $\sum_{i=1}^{3} \frac{\partial x'_i}{\partial x'_i}$.

3.5 Suppose $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ are basis vectors for a Cartesian coordinate system, and let $\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3$ be the images of $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ under a rotation. For each i, let

$$\mathbf{e}_i = a_{i1}\mathbf{e}'_1 + a_{i2}\mathbf{e}'_2 + a_{i3}\mathbf{e}'_3$$

be the expansion for $\mathbf{e}_i$ in terms of $\mathbf{e}'_j$. Find expressions for the a_{ij}'s in terms of $\mathbf{e}_i$ and $\mathbf{e}'_j$.

Solution: From $\mathbf{e}_i = a_{ij}\mathbf{e}'_j$, we claim that

$$a_{ij} = \mathbf{e}_i \cdot \mathbf{e}'_j.$$

We see this from considering the right-hand side:

$$\begin{aligned}\mathbf{e}_i \cdot \mathbf{e}'_j &= (a_{ik}\mathbf{e}'_k) \cdot \mathbf{e}'_j \\ &= a_{ik}(\mathbf{e}'_k \cdot \mathbf{e}'_j) \\ &= a_{ik}\delta_{kj} \quad \text{(since the basis vectors are orthonormal)} \\ &= a_{ij},\end{aligned}$$

as required.

3.6 Let $\mathbf{u}$ be the vector field defined by

$$\mathbf{u} = h(r)\mathbf{r},$$

where $h(r)$ is an arbitrary differentiable function, and $\mathbf{r}$ is the position vector $\mathbf{r} = (x_1, x_2, x_3)$ with $r = |\mathbf{r}|$.

Show, using suffix notation, that $\nabla \times \mathbf{u} = 0$.

[Hint: Exercise 2.8 can help you here: $\nabla h(r) = h'(r)\mathbf{r}/r$.]

Solution: In suffix notation, we have

$$\begin{aligned}
[\nabla \times \mathbf{u}]_i &= \epsilon_{ijk} \frac{\partial}{\partial x_j} u_k \\
&= \epsilon_{ijk} \frac{\partial}{\partial x_j} \big(h(r) x_k \big) \\
&= \epsilon_{ijk} \left[\frac{\partial h(r)}{\partial x_j} x_k + h(r) \frac{\partial x_k}{\partial x_j} \right] \\
&= \epsilon_{ijk} \left[\frac{h'(r) x_j}{r} x_k + h(r) \delta_{jk} \right] \qquad \left(\text{using Exercise 2.8: } \frac{\partial h(r)}{\partial r} = \frac{h'(r) x_j}{r} \right) \\
&= \frac{h'(r)}{r} \epsilon_{ijk} x_j x_k + h(r) \epsilon_{ijk} \delta_{jk} \\
&= \frac{h'(r)}{r} \epsilon_{ijk} x_j x_k \qquad \qquad (\text{as } \epsilon_{ijk} \delta_{jk} = 0)\,.
\end{aligned}$$

Now, notice that

$$\epsilon_{ijk} x_j x_k = -\epsilon_{jik} x_k x_j$$

as $\epsilon_{ijk} = -\epsilon_{ikj}$ but $x_j x_k = x_k x_j$. If we relabel $k \leftrightarrow j$, we get $-\epsilon_{ikj} x_k x_j = -\epsilon_{ijk} x_i x_j$. In conclusion,

$$\epsilon_{ijk} x_j x_k = -\epsilon_{ijk} x_j x_k \quad \text{thus } \epsilon_{ijk} x_j x_k = 0.$$

Hence

$$\nabla \times \mathbf{u} = \mathbf{0}.$$

3.7 Show that $\nabla \cdot \nabla^2 \mathbf{u} = \nabla^2 \nabla \cdot \mathbf{u}$ in two ways:

(*1*) directly using suffix notation;
(*2*) first using

$$\nabla^2 \mathbf{u} = \nabla(\nabla \cdot \mathbf{u}) - \nabla \times (\nabla \times \mathbf{u})$$

from the lectures, and then using suffix notation.

Solution:

(1) We have

$$\begin{aligned}
\nabla \cdot \nabla^2 \mathbf{u} &= \frac{\partial}{\partial x_j}\Big(\frac{\partial^2 u_j}{\partial x_k \partial x_k}\Big) \\
&= \frac{\partial^3 u_j}{\partial x_j \partial x_k \partial x_k} \\
&= \frac{\partial^3 u_j}{\partial x_k \partial x_k \partial x_j} \\
&= \frac{\partial^2}{\partial x_k \partial x_k}\Big(\frac{\partial u_j}{\partial x_j}\Big) \\
&= \nabla^2 \nabla \cdot \mathbf{u}.
\end{aligned}$$

(2) Here we have

$$\begin{aligned}
\nabla \cdot \nabla^2 \mathbf{u} &= \nabla \cdot \Big(\nabla(\nabla \cdot \mathbf{u}) - \nabla \times (\nabla \times \mathbf{u})\Big) \\
&= \frac{\partial}{\partial x_j}\Big(\nabla(\nabla \cdot \mathbf{u}) - \nabla \times (\nabla \times \mathbf{u})\Big)_j \\
&= \frac{\partial}{\partial x_j}\Big(\Big(\frac{\partial^2 u_k}{\partial x_j \partial x_k}\Big) - \epsilon_{j\ell m}\frac{\partial}{\partial x_\ell}(\nabla \times \mathbf{u})_m\Big) \\
&= \frac{\partial}{\partial x_j}\Big(\Big(\frac{\partial^2 u_k}{\partial x_j \partial x_k}\Big) - \epsilon_{j\ell m}\frac{\partial}{\partial x_\ell}\Big(\epsilon_{mpq}\frac{\partial u_q}{\partial x_p}\Big)\Big) \\
&= \frac{\partial}{\partial x_j}\Big(\Big(\frac{\partial^2 u_k}{\partial x_j \partial x_k}\Big) - \epsilon_{j\ell m}\epsilon_{mpq}\frac{\partial^2 u_q}{\partial x_\ell \partial x_p}\Big) \qquad \text{as } \frac{\partial \epsilon_{mpq}}{\partial x_\ell} = 0 \\
&= \frac{\partial^3 u_k}{\partial x_j \partial x_j \partial x_k} - (\delta_{jp}\delta_{\ell q} - \delta_{jq}\delta_{\ell p})\frac{\partial^3 u_q}{\partial x_j \partial x_\ell \partial x_p} \\
&= \frac{\partial^3 u_k}{\partial x_j \partial x_j \partial x_k} - \frac{\partial^3 u_\ell}{\partial x_j \partial x_\ell \partial x_j} + \frac{\partial^3 u_j}{\partial x_j \partial x_p \partial x_p} \\
&= \frac{\partial^3 u_k}{\partial x_j \partial x_j \partial x_k} - \frac{\partial^3 u_k}{\partial x_j \partial x_j \partial x_k} + \frac{\partial^3 u_k}{\partial x_j \partial x_j \partial x_k} \\
&= \frac{\partial^3 u_k}{\partial x_j \partial x_j \partial x_k},
\end{aligned}$$

as the order of derivatives does not matter, and as each of the three terms are separate we may relabel all dummy suffices. Hence we have the required result, as

$$\nabla^2 = \frac{\partial^2}{\partial x_j \partial x_j}.$$

3.8 Let f and g be scalar fields.

(*1*) Show, using suffix notation, that $\nabla \times (f\nabla f) = \mathbf{0}$.
(*2*) Simplify $\nabla \cdot (g\nabla g)$ to an expression involving just one operator acting on one scalar field.

Solution: **(1)** We have

$$\begin{aligned}[\nabla \times (f\nabla f)]_i &= \epsilon_{ijk}\frac{\partial (f\nabla f)_k}{\partial x_j} \\ &= \epsilon_{ijk}\frac{\partial}{\partial x_j}(f\frac{\partial f}{\partial x_k}) \\ &= \epsilon_{ijk}\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_k}\left(\frac{1}{2}f^2\right).\end{aligned}$$

As in the previous exercise, we see that

$$\epsilon_{ijk}\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_k}\left(\frac{1}{2}f^2\right) = -\epsilon_{ikj}\frac{\partial}{\partial x_k}\frac{\partial}{\partial x_j}\left(\frac{1}{2}f^2\right).$$

because the order of derivatives does not matter. Then, we relabel the indices $k \leftrightarrow j$ and get

$$\epsilon_{ijk}\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_k}\left(\frac{1}{2}f^2\right) = -\epsilon_{ijk}\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_k}\left(\frac{1}{2}f^2\right).$$

which implies $\epsilon_{ijk}\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_k}\left(\frac{1}{2}f^2\right) = (\mathbf{0})_i$.

(2) We have

$$\begin{aligned}\nabla \cdot (g\nabla g) &= \frac{\partial}{\partial x_j}(g\nabla g)_j \\ &= \frac{\partial}{\partial x_j}\left(g\frac{\partial g}{\partial x_j}\right) \\ &= \frac{\partial^2}{\partial x_j \partial x_j}\left(\frac{1}{2}g^2\right) \\ &= \nabla^2\Big(\frac{1}{2}g^2\Big).\end{aligned}$$
