Tensor Analysis – Practical 3

Information:

- Please make sure to complete **all** exercises **before** the next lecture.
- The exercises marked with [**See lecture**] were solved in class.
- The exercises are **not organised by difficulty**.

3.1 [**See lecture**] Use the **transformation law** to show that ∇ is a vector.

3.2 [**See lecture**] Let f be a scalar field. Show that $\nabla \cdot (\nabla f)$ is is a scalar using the **transformation law**.

3.3 Use the **transformation law** and the fact that ∇ and $\mathbf{w}$ are vectors to show that

$$\mathbf{w} \cdot \nabla$$

is a scalar.

3.4 Consider $L_{ij} = \frac{\partial x_i'}{\partial x_j}$. Show that $\frac{\partial L_{ij}}{\partial x_i'} = 0$.

3.5 Suppose $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ are basis vectors for a Cartesian coordinate system, and let $\mathbf{e}_1', \mathbf{e}_2', \mathbf{e}_3'$ be the images of $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ under a rotation. For each i, let

$$\mathbf{e}_i = a_{i1}\mathbf{e}_1' + a_{i2}\mathbf{e}_2' + a_{i3}\mathbf{e}_3'$$

be the expansion for $\mathbf{e}_i$ in terms of $\mathbf{e}_j'$. Find expressions for the a_{ij}'s in terms of $\mathbf{e}_i$ and $\mathbf{e}_j'$.

3.6 Let $\mathbf{u}$ be the vector field defined by

$$\mathbf{u} = h(r)\mathbf{r},$$

where $h(r)$ is an arbitrary differentiable function, and $\mathbf{r}$ is the position vector $\mathbf{r} = (x_1, x_2, x_3)$ with $r = |\mathbf{r}|$.

Show, using suffix notation, that $\nabla \times \mathbf{u} = 0$.

[Hint: Exercise 2.8 can help you here: $\nabla h(r) = h'(r)\mathbf{r}/r$.]

3.7 Show that $\nabla \cdot \nabla^2 \mathbf{u} = \nabla^2 \nabla \cdot \mathbf{u}$ in two ways:

(*1*) directly using suffix notation;
(*2*) first using

$$\nabla^2 \mathbf{u} = \nabla(\nabla \cdot \mathbf{u}) - \nabla \times (\nabla \times \mathbf{u})$$

from the lectures, and then using suffix notation.

3.8 Let f and g be scalar fields.

(1) Show, using suffix notation, that $\nabla \times (f \nabla f) = \mathbf{0}$.

(2) Simplify $\nabla \cdot (g \nabla g)$ to an expression involving just one operator acting on one scalar field.