Tensor Analysis – Practical 4

Solutions

Information:

- Please make sure to complete **all** exercises **before** the next lecture.
- The exercises marked with [**See lecture**] were solved in class.
- The exercises are **not organised by difficulty**.

4.1 Given an orthogonal coordinate system K with orthonormal basis $\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3$, consider the a new coordinate system K' with basis vectors

$$\begin{aligned}\mathbf{e}_1 &= \mathbf{i}_1 + \mathbf{i}_2 - \mathbf{i}_3,\\ \mathbf{e}_2 &= \mathbf{i}_1 + \mathbf{i}_2,\\ \mathbf{e}_3 &= \mathbf{i}_1 - \mathbf{i}_3.\end{aligned}$$

(1) Find a basis $\mathbf{e}^1, \mathbf{e}^2, \mathbf{e}^3$ dual to $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$.
(2) Find the covariant components of the vector joining the origin to the point $(2, 0, -1)$.
(3) Find the contravariant components of the vector joining the origin to the point $(2, 0, -1)$.

Solution: **(1)** We have

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \qquad \mathbf{e}_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \qquad \mathbf{e}_3 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}.$$

We will need

$$\begin{aligned}\mathbf{e}_1 \times \mathbf{e}_2 &= \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \times \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}\\ \mathbf{e}_2 \times \mathbf{e}_3 &= \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \times \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}\\ \mathbf{e}_3 \times \mathbf{e}_1 &= \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \times \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.\end{aligned}$$

We still need to compute V:

$$V = \mathbf{e}_1 \cdot (\mathbf{e}_2 \times \mathbf{e}_3) = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} = 1.$$

Applying to the dual basis formula, we get

$$\mathbf{e}^1 = \frac{\mathbf{e}_2 \times \mathbf{e}_3}{\mathbf{e}_1 \cdot (\mathbf{e}_2 \times \mathbf{e}_3)} = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}$$

$$\mathbf{e}^2 = \frac{\mathbf{e}_3 \times \mathbf{e}_1}{\mathbf{e}_2 \cdot (\mathbf{e}_3 \times \mathbf{e}_1)} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$\mathbf{e}^3 = \frac{\mathbf{e}_1 \times \mathbf{e}_2}{\mathbf{e}_3 \cdot (\mathbf{e}_1 \times \mathbf{e}_2)} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}.$$

Remark: When computing V, we can pick any of the three formulas:

$$V = \mathbf{e}_1 \cdot (\mathbf{e}_2 \times \mathbf{e}_3) = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} = 1,$$

$$V = \mathbf{e}_2 \cdot (\mathbf{e}_3 \times \mathbf{e}_1) = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = 1,$$

$$V = \mathbf{e}_3 \cdot (\mathbf{e}_1 \times \mathbf{e}_2) = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = 1.$$

(2) Let $\mathbf{A} = 2\mathbf{i}_1 - \mathbf{i}_3$. The **covariant** components of the vector $\mathbf{A}$ with respect to K' are the coefficients of the expansion

$$\mathbf{A} = A_1\mathbf{e}^1 + A_2\mathbf{e}^2 + A_3\mathbf{e}^3.$$

You can solve the system to find A_1, A_2, A_3, or you can use the formula $A_i = \mathbf{A} \cdot \mathbf{e}_i$. Let us show the second method:

$$A_1 = \mathbf{A} \cdot \mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = 2,$$

$$A_2 = \mathbf{A} \cdot \mathbf{e}_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = 1,$$

$$A_3 = \mathbf{A} \cdot \mathbf{e}_3 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = 2.$$

(3) Let $\mathbf{A} = 2\mathbf{i}_1 - \mathbf{i}_3$ again. The **contravariant** components of the vector $\mathbf{A}$ with respect to K' are the coefficients of the expansion

$$\mathbf{A} = A^1\mathbf{e}_1 + A^2\mathbf{e}_2 + A^3\mathbf{e}_3.$$

Again, you can either solve the system above to find A^1, A^2, A^3, or you can use the formula $A^i = \mathbf{A}\cdot\mathbf{e}^i$. We use the easier method:

$$A^1 = \mathbf{A}\cdot\mathbf{e}^1 = \begin{pmatrix}1\\0\\-1\end{pmatrix}\cdot\begin{pmatrix}-1\\1\\-1\end{pmatrix} = 0,$$
$$A^2 = \mathbf{A}\cdot\mathbf{e}^2 = \begin{pmatrix}1\\0\\-1\end{pmatrix}\cdot\begin{pmatrix}1\\0\\1\end{pmatrix} = 0,$$
$$A^3 = \mathbf{A}\cdot\mathbf{e}^3 = \begin{pmatrix}1\\0\\-1\end{pmatrix}\cdot\begin{pmatrix}1\\-1\\0\end{pmatrix} = 1.$$

4.2 [**See lecture**] Given an orthogonal coordinate system K with orthonormal basis $\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3$, consider the a new coordinate system K' with basis vectors

$$\mathbf{e}_1 = \mathbf{i}_1 + 2\mathbf{i}_2 + 4\mathbf{i}_3,$$
$$\mathbf{e}_2 = \mathbf{i}_2,$$
$$\mathbf{e}_3 = \mathbf{i}_1 + 2\mathbf{i}_2 + 5\mathbf{i}_3.$$

(1) Find a basis $\mathbf{e}^1, \mathbf{e}^2, \mathbf{e}^3$ dual to $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$.
(2) Find the covariant components of the vector joining the origin to the point $(1, 1, 1)$.
(3) Find the contravariant components of the vector joining the origin to the point $(1, 1, 1)$.

Solution:

(1) Here, we have

$$\mathbf{e}_1 = \begin{pmatrix}1\\2\\4\end{pmatrix}, \qquad \mathbf{e}_2 = \begin{pmatrix}0\\1\\0\end{pmatrix}, \qquad \mathbf{e}_3 = \begin{pmatrix}1\\2\\5\end{pmatrix}.$$

We will need

$$\mathbf{e}_1 \times \mathbf{e}_2 = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -4 \\ 0 \\ 1 \end{pmatrix}$$

$$\mathbf{e}_2 \times \mathbf{e}_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \times \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \\ -1 \end{pmatrix}$$

$$\mathbf{e}_3 \times \mathbf{e}_1 = \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix} \times \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}.$$

We still need to compute V:

$$V = \mathbf{e}_2 \cdot (\mathbf{e}_3 \times \mathbf{e}_1) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} = 1.$$

Applying to the dual basis formula, we get

$$\mathbf{e}^1 = \frac{\mathbf{e}_2 \times \mathbf{e}_3}{\mathbf{e}_1 \cdot (\mathbf{e}_2 \times \mathbf{e}_3)} = \mathbf{e}_2 \times \mathbf{e}_3 = \begin{pmatrix} 5 \\ 0 \\ -1 \end{pmatrix}$$

$$\mathbf{e}^2 = \frac{\mathbf{e}_3 \times \mathbf{e}_1}{\mathbf{e}_2 \cdot (\mathbf{e}_3 \times \mathbf{e}_1)} = \mathbf{e}_3 \times \mathbf{e}_1 = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$$

$$\mathbf{e}^3 = \frac{\mathbf{e}_1 \times \mathbf{e}_2}{\mathbf{e}_3 \cdot (\mathbf{e}_1 \times \mathbf{e}_2)} = \mathbf{e}_1 \times \mathbf{e}_2 = \begin{pmatrix} -4 \\ 0 \\ 1 \end{pmatrix}.$$

(2) Let $\mathbf{B} = \mathbf{i}_1 + \mathbf{i}_2 + \mathbf{i}_3$. The **covariant** components of the vector $\mathbf{B}$ with respect to K' are the coefficients of the expansion

$$\mathbf{B} = B_1\mathbf{e}^1 + B_2\mathbf{e}^2 + B_3\mathbf{e}^3.$$

They are given by the formula $B_i = \mathbf{B} \cdot \mathbf{e}_i$:

$$B_1 = \mathbf{B} \cdot \mathbf{e}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} = 7,$$

$$B_2 = \mathbf{B} \cdot \mathbf{e}_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = 1,$$

$$B_3 = \mathbf{B} \cdot \mathbf{e}_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix} = 8.$$

(3) Let $\mathbf{B} = \mathbf{i}_1 + \mathbf{i}_2 + \mathbf{i}_3$. The **contravariant** components of the vector $\mathbf{B}$ with respect to K' are the coefficients of the expansion

$$\mathbf{B} = B^1\mathbf{e}_1 + B^2\mathbf{e}_2 + B^3\mathbf{e}_3,$$

which are given by the formula $A^i = \mathbf{A} \cdot \mathbf{e}^i$:

$$B^1 = \mathbf{B} \cdot \mathbf{e}^1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 5 \\ 0 \\ -1 \end{pmatrix} = 4,$$

$$B^2 = \mathbf{B} \cdot \mathbf{e}^2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} = -1,$$

$$B^3 = \mathbf{B} \cdot \mathbf{e}^3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} -4 \\ 0 \\ 1 \end{pmatrix} = -3.$$

4.3 Given an orthogonal coordinate system K with orthonormal basis $\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3$, consider the a new coordinate system K' with basis vectors

$$\mathbf{e}_1 = \mathbf{i}_1 + 2\mathbf{i}_2,$$
$$\mathbf{e}_2 = 2\mathbf{i}_1 + 2\mathbf{i}_2 + 4\mathbf{i}_3,$$
$$\mathbf{e}_3 = 2\mathbf{i}_3.$$

(1) Find a basis $\mathbf{e}^1, \mathbf{e}^2, \mathbf{e}^3$ dual to $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$.
(2) Find the covariant components of the vector $\mathbf{C} = 2\mathbf{i}_2 + 2\mathbf{i}_3$.
(3) Find the contravariant components of the vector $\mathbf{C} = 2\mathbf{i}_2 + 2\mathbf{i}_3$.

Solution:

(1) Here, we have

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \qquad \mathbf{e}_2 = \begin{pmatrix} 2 \\ 2 \\ 4 \end{pmatrix}, \qquad \mathbf{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}.$$

We will need

$$\mathbf{e}_1 \times \mathbf{e}_2 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \times \begin{pmatrix} 2 \\ 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 8 \\ -4 \\ -2 \end{pmatrix}$$
$$\mathbf{e}_2 \times \mathbf{e}_3 = \begin{pmatrix} 2 \\ 2 \\ 4 \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 \\ -4 \\ 0 \end{pmatrix}$$
$$\mathbf{e}_3 \times \mathbf{e}_1 = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} \times \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} -4 \\ 2 \\ 0 \end{pmatrix}.$$

We also need:

$$V = \mathbf{e}_3 \cdot (\mathbf{e}_1 \times \mathbf{e}_2) = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 8 \\ -4 \\ -2 \end{pmatrix} = -4.$$

Applying to the dual basis formula, we get

$$\mathbf{e}^1 = \frac{\mathbf{e}_2 \times \mathbf{e}_3}{\mathbf{e}_1 \cdot (\mathbf{e}_2 \times \mathbf{e}_3)} = \frac{\mathbf{e}_2 \times \mathbf{e}_3}{-4} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$
$$\mathbf{e}^2 = \frac{\mathbf{e}_3 \times \mathbf{e}_1}{\mathbf{e}_2 \cdot (\mathbf{e}_3 \times \mathbf{e}_1)} = \frac{\mathbf{e}_3 \times \mathbf{e}_1}{-4} = \begin{pmatrix} 1 \\ -\frac{1}{2} \\ 0 \end{pmatrix}$$
$$\mathbf{e}^3 = \frac{\mathbf{e}_1 \times \mathbf{e}_2}{\mathbf{e}_3 \cdot (\mathbf{e}_1 \times \mathbf{e}_2)} = \frac{\mathbf{e}_1 \times \mathbf{e}_2}{-4} = \begin{pmatrix} -2 \\ 1 \\ \frac{1}{2} \end{pmatrix}.$$

(2) The **covariant** components of the vector $\mathbf{C}$ with respect to K' are the coefficients of the expansion

$$\mathbf{C} = C_1 \mathbf{e}^1 + C_2 \mathbf{e}^2 + C_3 \mathbf{e}^3.$$

They are given by the formula $C_i = \mathbf{C} \cdot \mathbf{e}_i$:

$$C_1 = \mathbf{C}\cdot\mathbf{e}_1 = \begin{pmatrix}0\\2\\2\end{pmatrix}\cdot\begin{pmatrix}1\\2\\0\end{pmatrix} = 4,$$

$$C_2 = \mathbf{C}\cdot\mathbf{e}_2 = \begin{pmatrix}0\\2\\2\end{pmatrix}\cdot\begin{pmatrix}2\\2\\4\end{pmatrix} = 12,$$

$$C_3 = \mathbf{C}\cdot\mathbf{e}_3 = \begin{pmatrix}0\\2\\2\end{pmatrix}\cdot\begin{pmatrix}0\\0\\2\end{pmatrix} = 4.$$

(3) The **contravariant** components of the vector $\mathbf{C}$ with respect to K' are the coefficients of the expansion

$$\mathbf{C} = C^1\mathbf{e}_1 + C^2\mathbf{e}_2 + C^3\mathbf{e}_3,$$

which are given by the formula $A^i = \mathbf{A}\cdot\mathbf{e}^i$:

$$C^1 = \mathbf{C}\cdot\mathbf{e}^1 = \begin{pmatrix}0\\2\\2\end{pmatrix}\cdot\begin{pmatrix}-1\\1\\0\end{pmatrix} = 2,$$

$$C^2 = \mathbf{C}\cdot\mathbf{e}^2 = \begin{pmatrix}0\\2\\2\end{pmatrix}\cdot\begin{pmatrix}1\\-\frac{1}{2}\\0\end{pmatrix} = -1,$$

$$C^3 = \mathbf{C}\cdot\mathbf{e}^3 = \begin{pmatrix}0\\2\\2\end{pmatrix}\cdot\begin{pmatrix}-2\\1\\\frac{1}{2}\end{pmatrix} = 3.$$

4.4 Given an orthogonal coordinate system K with orthonormal basis $\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3$, consider the vector $\mathbf{B}$ with components

$$\mathbf{B} = \mathbf{i}_1 + 3\mathbf{i}_2.$$

Let K be a new coordinate system with basis vectors

$$\mathbf{e}'_1 = \mathbf{i}_1,$$
$$\mathbf{e}'_2 = \mathbf{i}_1 - \mathbf{i}_2,$$
$$\mathbf{e}'_3 = \mathbf{i}_1 + \mathbf{i}_2 + \mathbf{i}_3.$$

(*1*) Work out the expansion coefficients $L^1_{i'}, L^2_{i'}, L^3_{i'}$ for each $i = 1, 2, 3$.
(*2*) Compute the covariant components B'_i of $\mathbf{B}$ in the coordinate system K'.
(*3*) Find a basis dual to $\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3$.
(*4*) Compute the contravariant components B'^i of $\mathbf{B}$ in the coordinate system K'.

(5) Let $g^{ik} = \mathbf{e}'^i \cdot \mathbf{e}'^k$. Compute these g^{ik} for all $i, k = 1, 2, 3$. (It might be more convenient to write your answers as the entries of a matrix.)

(6) Using your answers to parts (2) and (5), work out the contravariant components B'^i of $\mathbf{B}$ in K' in another way.

Solution: **(1)** We $L^n_{m'} = \mathbf{e}'_m \cdot \mathbf{i}_n$, so

$$\begin{aligned} L^1_{1'} &= 1, \quad L^2_{1'} = 0, \quad L^3_{1'} = 0, \\ L^1_{2'} &= 1, \quad L^2_{2'} = -1, \quad L^3_{2'} = 0, \\ L^1_{3'} &= 1, \quad L^2_{3'} = 1, \quad L^3_{3'} = 1. \end{aligned}$$

(2) Hence, from $B'_i = L^j_{i'} B_j$ we obtain

$$\begin{aligned} B'_1 &= L^1_{1'} B_1 + L^2_{1'} B_2 + L^3_{1'} B_3 = 1, \\ B'_2 &= L^1_{2'} B_1 + L^2_{2'} B_2 + L^3_{2'} B_3 = -2, \\ B'_3 &= L^1_{3'} B_1 + L^2_{3'} B_2 + L^3_{3'} B_3 = 4. \end{aligned}$$

(3) We have

$$\mathbf{e}'_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad \mathbf{e}'_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \qquad \mathbf{e}'_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

We will need

$$\begin{aligned} \mathbf{e}'_1 \times \mathbf{e}'_2 &= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \times \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} \\ \mathbf{e}'_2 \times \mathbf{e}'_3 &= \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \times \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix} \\ \mathbf{e}'_3 \times \mathbf{e}'_1 &= \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \times \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}. \end{aligned}$$

Now

$$\mathbf{e}'^1 = \frac{\mathbf{e}'_2 \times \mathbf{e}'_3}{\mathbf{e}'_1 \cdot (\mathbf{e}'_2 \times \mathbf{e}'_3)} = -\begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix},$$

$$\mathbf{e}'^2 = \frac{\mathbf{e}'_3 \times \mathbf{e}'_1}{\mathbf{e}'_2 \cdot (\mathbf{e}'_3 \times \mathbf{e}'_1)} = -\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix},$$

$$\mathbf{e}'^3 = \frac{\mathbf{e}'_1 \times \mathbf{e}'_2}{\mathbf{e}'_3 \cdot (\mathbf{e}'_1 \times \mathbf{e}'_2)} = -\begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

gives us the dual basis.

(4) We have $L_n^{m'} = \mathbf{e}'^m \cdot \mathbf{i}_n$, thus

$$L_1^{1'} = 1, \quad L_2^{1'} = 1, \quad L_3^{1'} = -2,$$
$$L_1^{2'} = 0, \quad L_2^{2'} = -1, \quad L_3^{2'} = 1,$$
$$L_1^{3'} = 0, \quad L_2^{3'} = 0, \quad L_3^{3'} = 1.$$

Hence, from $B'^i = L_j^{i'} B^j$ we obtain

$$B'^1 = L_1^{1'} B^1 + L_2^{1'} B^2 + L_3^{1'} B^3 = 4,$$
$$B'^2 = L_1^{2'} B^1 + L_2^{2'} B^2 + L_3^{2'} B^3 = -3,$$
$$B'^3 = L_1^{3'} B^1 + L_2^{3'} B^2 + L_3^{3'} B^3 = 0.$$

(5) It is easier to write g^{ik} as a matrix:

$$g^{ik} = \begin{pmatrix} 6 & -3 & -2 \\ -3 & 2 & 1 \\ -2 & 1 & 1 \end{pmatrix}.$$

(6) We will use the relation $B'^i = g^{ik} B'_k = \sum_{k=1}^{3} g^{ik} B'_k$. Performing this straightforward computation yields the same answers as in part (4).

4.5 [Question from Final Exam 23-24] In this question, denote by K the Cartesian coordinate system with vector basis $\mathbf{i}_1$, $\mathbf{i}_2$, $\mathbf{i}_3$, the standard orthonormal basis of $\mathbb{R}^3$. Denote by K' the coordinate system with vector basis $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ given by

$$\mathbf{e}_1 = 2\mathbf{i}_1 + \mathbf{i}_3$$
$$\mathbf{e}_2 = \mathbf{i}_2$$
$$\mathbf{e}_3 = \mathbf{i}_1 - \mathbf{i}_2 + \mathbf{i}_3.$$

(1) Find the dual basis $\mathbf{e}^1$, $\mathbf{e}^2$, $\mathbf{e}^3$.

(2) Find the covariant and contravariant components of the vector $\mathbf{V} = 3\mathbf{i}_1 - 2\mathbf{i}_2 + \mathbf{i}_3$ with respect to the bases $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ and $\mathbf{e}^1$, $\mathbf{e}^2$, $\mathbf{e}^3$.

Solution:

(a) We have

$$\mathbf{e}_1 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \quad \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \mathbf{e}_3 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}.$$

We will need

$$\begin{aligned}
\mathbf{e}_1 \times \mathbf{e}_2 &= \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} \\
\mathbf{e}_2 \times \mathbf{e}_3 &= \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \times \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \\
\mathbf{e}_3 \times \mathbf{e}_1 &= \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \times \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}.
\end{aligned}$$

Notice that

$$\mathbf{e}_1 \cdot (\mathbf{e}_2 \times \mathbf{e}_3) = \mathbf{e}_2 \cdot (\mathbf{e}_3 \times \mathbf{e}_1) = \mathbf{e}_3 \cdot (\mathbf{e}_1 \times \mathbf{e}_2) = 1.$$

Consequently

$$\begin{aligned}
\mathbf{e}^1 &= \frac{\mathbf{e}_2 \times \mathbf{e}_3}{\mathbf{e}_1 \cdot (\mathbf{e}_2 \times \mathbf{e}_3)} = \mathbf{e}_2 \times \mathbf{e}_3 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \\
\mathbf{e}^2 &= \frac{\mathbf{e}_3 \times \mathbf{e}_1}{\mathbf{e}_2 \cdot (\mathbf{e}_3 \times \mathbf{e}_1)} = \mathbf{e}_3 \times \mathbf{e}_1 = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} \\
\mathbf{e}^3 &= \frac{\mathbf{e}_1 \times \mathbf{e}_2}{\mathbf{e}_3 \cdot (\mathbf{e}_1 \times \mathbf{e}_2)} = \mathbf{e}_1 \times \mathbf{e}_2 = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}.
\end{aligned}$$

(b) The covariant components of $\mathbf{V}$ are given by

$$V_i = \mathbf{V} \cdot \mathbf{e}_i.$$

Thus

$$V_1 = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} = 7, \quad V_2 = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = -2, \quad V_3 = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = 6.$$

Similarly, the contravariant components of $\mathbf{V}$ are given by

$$V^i = \mathbf{V} \cdot \mathbf{e}^i.$$

Thus

$$V_1 = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = 2, \quad V_2 = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} = -3, \quad V_3 = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} = -1.$$

4.6 Consider dual bases $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ and $\mathbf{e}^1, \mathbf{e}^2, \mathbf{e}^3$, and denote

$$V = \mathbf{e}_1 \cdot (\mathbf{e}_2 \times \mathbf{e}_3), \quad \text{and} \quad V' = \mathbf{e}^1 \cdot (\mathbf{e}^2 \times \mathbf{e}^3).$$

Show, using the vector triple product formula, that $V' = 1/V$.

Solution: In the lectures, we have shown

$$V' = \mathbf{e}^1 \cdot (\mathbf{e}_2 \times \mathbf{e}_3).$$

As

$$\mathbf{e}^1 = \frac{\mathbf{e}_2 \times \mathbf{e}_3}{V}, \qquad \mathbf{e}^2 = \frac{\mathbf{e}_3 \times \mathbf{e}_1}{V}, \qquad \mathbf{e}^3 = \frac{\mathbf{e}_1 \times \mathbf{e}_2}{V},$$

we have

$$\begin{aligned}
V' &= \frac{\mathbf{e}_2 \times \mathbf{e}_3}{V} \cdot \left(\frac{\mathbf{e}_3 \times \mathbf{e}_1}{V} \times \frac{\mathbf{e}_1 \times \mathbf{e}_2}{V} \right) \\
&= \frac{1}{V^3} (\mathbf{e}_2 \times \mathbf{e}_3) \cdot (((\mathbf{e}_3 \times \mathbf{e}_1) \cdot \mathbf{e}_2)\mathbf{e}_1 - ((\mathbf{e}_3 \times \mathbf{e}_1) \cdot \mathbf{e}_1)\mathbf{e}_2) \\
&= \frac{1}{V^3} (\mathbf{e}_2 \times \mathbf{e}_3) \cdot ((\mathbf{e}_3 \times \mathbf{e}_1) \cdot \mathbf{e}_2)\mathbf{e}_1) \qquad \text{(because } (\mathbf{e}_3 \times \mathbf{e}_1) \perp \mathbf{e}_1) \\
&= \frac{1}{V^3} (\mathbf{e}_2 \times \mathbf{e}_3) \cdot (V \mathbf{e}_1) \\
&= \frac{1}{V^2} (\mathbf{e}_2 \times \mathbf{e}_3) \cdot \mathbf{e}_1 \\
&= \frac{1}{V},
\end{aligned}$$

as required.

4.7 **[See lecture]** Express the scalar triple product $(\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C}$ in terms of the covariant and contravariant components of the vectors $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$, with respect to dual bases. You may use the equalities

(1) $$\mathbf{A} \times \mathbf{B} = (A^j B^k - A^k B^j)V = \frac{1}{V}(A_j B_k - A_k B_j).$$

Solution: Writing $\mathbf{D} = \mathbf{A} \times \mathbf{B}$, we have from Example 3.8 that,

$$\begin{aligned}
\mathbf{D} = D_i \mathbf{e}^i &= (A^j B^k - A^k B^j) V \mathbf{e}^i \\
&= D^i \mathbf{e}_i = \frac{1}{V}(A_j B_k - A_k B_j)\mathbf{e}_i.
\end{aligned}$$

Thus, writing $\mathbf{E} = \mathbf{D} \cdot \mathbf{C}$, we obtain

$$\mathbf{E} = D^i \mathbf{e}_i \cdot C^\ell \mathbf{e}_\ell = \frac{g_{i\ell}}{V}(A_j B_k - A_k B_j) C^\ell,$$

as $g_{i\ell} = \mathbf{e}_i \cdot \mathbf{e}_\ell$. Similarly,

$$\mathbf{E} = D_i \mathbf{e}^i \cdot C_\ell \mathbf{e}^\ell = g^{i\ell} V (A^j B^k - A^k B^j) C_\ell,$$

$$\mathbf{E} = D^i \mathbf{e}_i \cdot C_\ell \mathbf{e}^\ell = \frac{1}{V} (A_j B_k - A_k B_j) C_\ell,$$

and

$$\mathbf{E} = D_i \mathbf{e}^i \cdot C^\ell \mathbf{e}_\ell = V (A^j B^k - A^k B^j) C^\ell.$$