

Tensor Analysis – Practical 4

Information:

- Please make sure to complete **all** exercises **before** the next lecture.
- The exercises marked with [See lecture] were solved in class.
- The exercises are **not organised by difficulty**.

4.1 Given an orthogonal coordinate system K with orthonormal basis $\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3$, consider the a new coordinate system K' with basis vectors

$$\mathbf{e}_1 = \mathbf{i}_1 + \mathbf{i}_2 - \mathbf{i}_3,$$

$$\mathbf{e}_2 = \mathbf{i}_1 + \mathbf{i}_2,$$

$$\mathbf{e}_3 = \mathbf{i}_1 - \mathbf{i}_3.$$

- (1) Find a basis $\mathbf{e}^1, \mathbf{e}^2, \mathbf{e}^3$ dual to $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$.
- (2) Find the covariant components of the vector joining the origin to the point $(2, 0, -1)$.
- (3) Find the contravariant components of the vector joining the origin to the point $(2, 0, -1)$.

4.2 [See lecture] Given an orthogonal coordinate system K with orthonormal basis $\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3$, consider the a new coordinate system K' with basis vectors

$$\mathbf{e}_1 = \mathbf{i}_1 + 2\mathbf{i}_2 + 4\mathbf{i}_3,$$

$$\mathbf{e}_2 = \mathbf{i}_2,$$

$$\mathbf{e}_3 = \mathbf{i}_1 + 2\mathbf{i}_2 + 5\mathbf{i}_3.$$

- (1) Find a basis $\mathbf{e}^1, \mathbf{e}^2, \mathbf{e}^3$ dual to $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$.
- (2) Find the covariant components of the vector joining the origin to the point $(1, 1, 1)$.
- (3) Find the contravariant components of the vector joining the origin to the point $(1, 1, 1)$.

4.3 Given an orthogonal coordinate system K with orthonormal basis $\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3$, consider the a new coordinate system K' with basis vectors

$$\mathbf{e}_1 = \mathbf{i}_1 + 2\mathbf{i}_2,$$

$$\mathbf{e}_2 = 2\mathbf{i}_1 + 2\mathbf{i}_2 + 4\mathbf{i}_3,$$

$$\mathbf{e}_3 = 2\mathbf{i}_3.$$

- (1) Find a basis $\mathbf{e}^1, \mathbf{e}^2, \mathbf{e}^3$ dual to $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$.
- (2) Find the covariant components of the vector $\mathbf{C} = 2\mathbf{i}_2 + 2\mathbf{i}_3$.
- (3) Find the contravariant components of the vector $\mathbf{C} = 2\mathbf{i}_2 + 2\mathbf{i}_3$.

4.4 Given an orthogonal coordinate system K with orthonormal basis $\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3$, consider the vector \mathbf{B} with components

$$\mathbf{B} = \mathbf{i}_1 + 3\mathbf{i}_2.$$

Let K' be a new coordinate system with basis vectors

$$\begin{aligned}\mathbf{e}'_1 &= \mathbf{i}_1, \\ \mathbf{e}'_2 &= \mathbf{i}_1 - \mathbf{i}_2, \\ \mathbf{e}'_3 &= \mathbf{i}_1 + \mathbf{i}_2 + \mathbf{i}_3.\end{aligned}$$

- (1) Work out the expansion coefficients $L'_{i'}^1, L'_{i'}^2, L'_{i'}^3$ for each $i = 1, 2, 3$.
- (2) Compute the covariant components $B'_{i'}$ of \mathbf{B} in the coordinate system K' .
- (3) Find a basis dual to $\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3$.
- (4) Compute the contravariant components B'^i of \mathbf{B} in the coordinate system K' .
- (5) Let $g^{ik} = \mathbf{e}'^i \cdot \mathbf{e}'^k$. Compute these g^{ik} for all $i, k = 1, 2, 3$. (It might be more convenient to write your answers as the entries of a matrix.)
- (6) Using your answers to parts (2) and (5), work out the contravariant components B'^i of \mathbf{B} in K' in another way.

4.5 [Question from Final Exam 23-24] In this question, denote by K the Cartesian coordinate system with vector basis $\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3$, the standard orthonormal basis of \mathbb{R}^3 . Denote by K' the coordinate system with vector basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ given by

$$\begin{aligned}\mathbf{e}_1 &= 2\mathbf{i}_1 + \mathbf{i}_3 \\ \mathbf{e}_2 &= \mathbf{i}_2 \\ \mathbf{e}_3 &= \mathbf{i}_1 - \mathbf{i}_2 + \mathbf{i}_3.\end{aligned}$$

- (1) Find the dual basis $\mathbf{e}^1, \mathbf{e}^2, \mathbf{e}^3$.
- (2) Find the covariant and contravariant components of the vector $\mathbf{V} = 3\mathbf{i}_1 - 2\mathbf{i}_2 + \mathbf{i}_3$ with respect to the bases $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ and $\mathbf{e}^1, \mathbf{e}^2, \mathbf{e}^3$.

4.6 Consider dual bases $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ and $\mathbf{e}^1, \mathbf{e}^2, \mathbf{e}^3$, and denote

$$V = \mathbf{e}_1 \cdot (\mathbf{e}_2 \times \mathbf{e}_3), \quad \text{and} \quad V' = \mathbf{e}^1 \cdot (\mathbf{e}^2 \times \mathbf{e}^3).$$

Show, using the vector triple product formula, that $V' = 1/V$.

4.7 [See lecture] Express the scalar triple product $(\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C}$ in terms of the covariant and contravariant components of the vectors \mathbf{A}, \mathbf{B} and \mathbf{C} , with respect to dual bases. You may use the equalities

$$(1) \quad \mathbf{A} \times \mathbf{B} = (A^j B^k - A^k B^j) \mathbf{V} = \frac{1}{V} (A_j B_k - A_k B_j).$$