Tensor Analysis – Practical 5

Solutions

Information:

- Please make sure to complete **all** exercises **before** the next lecture.
- The exercises marked with **[See lecture]** were solved in class.
- The exercises are **not organised by difficulty**.

5.1 [**See Lecture**] Write down the transformation rule for:

(*1*) a tensor of rank four,
(*2*) a tensor of rank six.

Solution: Rank four: $T_{ijk\ell} = L_{ia}L_{jb}L_{kc}L_{\ell d}T_{abcd}$.

Rank six: $R_{ijk\ell mn} = L_{ia}L_{jb}L_{kc}L_{\ell d}L_{me}L_{nf}R_{abcdef}$.

5.2 If $Q_{ijk\ell}$ is a tensor of rank four, show that $Q_{ijj\ell}$ is a tensor of rank two.

Solution: As $Q_{ijk\ell}$ is a tensor of rank four, it satisfies

$$Q'_{ijk\ell} = L_{im}L_{jn}L_{kp}L_{\ell q}Q_{mnpq}.$$

Hence

$$\begin{aligned} Q'_{ijj\ell} &= L_{im}L_{jn}L_{jp}L_{\ell q}Q_{mnpq} \\ &= L_{im}\delta_{np}L_{\ell q}Q_{mnpq} \qquad (\text{ using } L_{jn}L_{jp} = \delta_{np}) \\ &= L_{im}L_{\ell q}Q_{mnnq}, \end{aligned}$$

which gives the result.

5.3 **[See lecture]** If T_{ij} is a tensor, show that T_{ii} is a scalar.

Solution: Since T_{ij} is a tensor, we know that $T'_{ij} = L_{ik}L_{jm}T_{km}$. Therefore

$$\begin{aligned} T'_{ii} &= L_{ik}L_{im}T_{km} \\ &= \delta_{km}T_{km} \\ &= T_{mm} \\ &= T_{ii}, \end{aligned}$$

where the last equality comes from the fact that we can replace a dummy index with another dummy index.

5.4 Consider the $2D$ polar coordinates $(x^1, x^2) = (r, \theta)$. In this system, the position vector $\mathbf{r}$ is given by

$$\mathbf{r} = r\cos\theta\,\mathbf{i}_1 + r\sin\theta\,\mathbf{i}_2,$$

where $\mathbf{i}_1$, $\mathbf{i}_2$ are the usual 2-dimensional Cartesian basis vectors.

(1) Find the corresponding basis vectors $\mathbf{e}_1$ and $\mathbf{e}_2$.

(2) Find the covariant metric tensor g_{ij} and give your answer as a 2×2 matrix. Using the fact that the basis $\mathbf{e}_1, \mathbf{e}_2$ is orthogonal, find the contravariant metric tensor g^{ij}.

(3) Give an expression for the arc length of the polar coordinate system, plus explicitly state the metric coefficients h_1, h_2.

Solution: **(1)** Remember that $\mathbf{e}_i = \frac{\partial \mathbf{r}}{\partial x^i}$. Thus,

$$\begin{aligned} \mathbf{e}_1 &= \frac{\partial \mathbf{r}}{\partial x^1} = \frac{\partial \mathbf{r}}{\partial r} = \cos\theta\,\mathbf{i}_1 + \sin\theta\,\mathbf{i}_2, \\ \mathbf{e}_2 &= \frac{\partial \mathbf{r}}{\partial x^2} = \frac{\partial \mathbf{r}}{\partial \theta} = -r\sin\theta\,\mathbf{i}_1 + r\cos\theta\,\mathbf{i}_2. \end{aligned}$$

(2) Using the result in (1), we obtain

$$\begin{aligned} g_{11} &= \mathbf{e}_1 \cdot \mathbf{e}_1 = \cos^2\theta + \sin^2\theta = 1, \\ g_{22} &= \mathbf{e}_2 \cdot \mathbf{e}_2 = r^2\sin^2\theta + r^2\sin^2\theta = r^2, \\ g_{12} &= \mathbf{e}_1 \cdot \mathbf{e}_2 = g_{21} = \mathbf{e}_2 \cdot \mathbf{e}_1 = 0. \end{aligned}$$

In matrix form

$$g_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}.$$

Using orthogonality, we have $g_{ii} = 1/g^{ii}$ for each $i = 1, 2$. Thus

$$g^{ij} = \begin{pmatrix} 1 & 0 \\ 0 & 1/r^2 \end{pmatrix}$$

(3) The element of the arc length is

$$\begin{aligned}(ds)^2 &= g_{ij}dx^i dx^j = g_{11}dx^1dx^1 + g_{22}dx^2dx^2 \\ &= (dr)^2 + (rd\theta)^2.\end{aligned}$$

Hence the metric coefficients are

$$h_1 = 1, \quad h_2 = r.$$

5.5 In spherical coordinates $(x^1, x^2, x^3) = (r, \phi, \theta)$, the position vector $\mathbf{r}$ is given by

$$\mathbf{r} = r\sin\phi\cos\theta\,\mathbf{i}_1 + r\sin\phi\sin\theta\,\mathbf{i}_2 + r\cos\phi\,\mathbf{i}_3.$$

Find the basis vectors and the metric coefficients.

Solution: We have

$$\begin{aligned}\mathbf{e}_1 &= \frac{\partial\mathbf{r}}{\partial r} = \sin\phi\cos\theta\,\mathbf{i}_1 + \sin\phi\sin\theta\,\mathbf{i}_2 + \cos\phi\,\mathbf{i}_3 \\ \mathbf{e}_2 &= \frac{\partial\mathbf{r}}{\partial\phi} = r\cos\phi\cos\theta\,\mathbf{i}_1 + r\cos\phi\sin\theta\,\mathbf{i}_2 - r\sin\phi\,\mathbf{i}_3 \\ \mathbf{e}_3 &= \frac{\partial\mathbf{r}}{\partial\theta} = -r\sin\phi\sin\theta\mathbf{i}_1 + r\sin\phi\cos\theta\,\mathbf{i}_2.\end{aligned}$$

Hence the element of the arc length is

$$(ds)^2 = (dr)^2 + (rd\phi)^2 + (r\sin\phi\,d\theta)^2$$

The metric coefficients are then

$$h_1 = 1, \quad h_2 = r, \quad h_3 = r\sin\phi.$$

5.6 **[See Lecture]** Given two vectors $\mathbf{A}$ and $\mathbf{B}$, with components A_i and B_i, show that the second-order tensor with components

$$C_{ik} = A_i B_k - A_k B_i$$

is antisymmetric.

Solution: We must show

$$C_{ik} = -C_{ki}.$$

We have

$$C_{ki} = A_k B_i - A_i B_k = -(A_i B_k - A_k B_i) = -C_{ik}.$$

5.7 **[Question from the Main Exam 23-24]** The third-rank tensor T_{ijk} is symmetric with respect to its last two suffixes but antisymmetric with respect to its first and second suffices. Show that all entries of this tensor are zero (i.e. $T_{ijk} = 0$, for all choices of i,j,k).

Solution: One the one hand,

$$T_{ijk} = -T_{jik} = -T_{jki}.$$

On the other hand,

$$T_{ijk} = T_{ikj} = -T_{kij} = -T_{kji} = -(-T_{jki}) = T_{jki}.$$

Consequently, $-T_{jki} = T_{jki}$, and therefore $T_{jki} = 0$, as required.

5.8 Let q^1, q^2, q^3 be coordinates related to orthogonal coordinates x_1, x_2, x_3 with orthonormal basis $\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3$ by the formulae

$$q^1 = x_1 + x_2, \quad q^2 = x_1 - x_2, \quad q^3 = 2x_3.$$

(*1*) Find the corresponding basis vectors and show that they form an orthogonal basis. [Hint: Using $\mathbf{r} = x_1\mathbf{i}_1 + x_2\mathbf{i}_2 + x_3\mathbf{i}_3$, you need to find expressions for x_1, x_2, x_3 in terms of q^1, q^2, q^3 and substitute these into your expansion for $\mathbf{r}$.]

(*2*) Find the metric tensor g_{ik} and give your answer as a matrix. Using the fact that the basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ is orthogonal, find the contravariant metric tensor g^{ik}.

(*3*) Give an expression for the arc length of the coordinate system with respect to the basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$.

(*4*) Find the covariant and contravariant components of the vectors

$$\mathbf{A} = 2\mathbf{i}_1, \quad \mathbf{B} = \mathbf{i}_1 + \mathbf{i}_2, \quad \mathbf{C} = 2\mathbf{i}_1 - 3\mathbf{i}_3,$$

with respect to the basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$, but without computing the dual basis $\mathbf{e}^1, \mathbf{e}^2, \mathbf{e}^3$.

Solution:

(1) We have the position vector $\mathbf{r} = x_1\mathbf{i}_1 + x_2\mathbf{i}_2 + x_3\mathbf{i}_3$ is given by

$$\mathbf{r} = \frac{q^1 + q^2}{2}\mathbf{i}_1 + \frac{q^1 - q^2}{2}\mathbf{i}_2 + \frac{q^3}{2}\mathbf{i}_3.$$

Hence the basis vectors are given by

$$\mathbf{e}_1 = \frac{\partial \mathbf{r}}{\partial q^1} = \frac{\mathbf{i}_1}{2} + \frac{\mathbf{i}_2}{2},$$
$$\mathbf{e}_2 = \frac{\partial \mathbf{r}}{\partial q^2} = \frac{\mathbf{i}_1}{2} - \frac{\mathbf{i}_2}{2},$$
$$\mathbf{e}_3 = \frac{\partial \mathbf{r}}{\partial q^3} = \frac{\mathbf{i}_3}{2}.$$

Clearly one has $\mathbf{e}_i \cdot \mathbf{e}_k = 0$ when $i \neq k$, hence the basis is orthogonal.

(2) The metric tensor g_{ik} is given by

$$[g_{ik}] = \begin{pmatrix} 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/4 \end{pmatrix}.$$

Now as we are dealing with an orthogonal basis, we have $g^{ii} = 1/g_{ii}$. Thus

$$[g^{ik}] = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}.$$

(3) As the basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ is orthogonal the formula for the arc length is

$$(ds)^2 = g_{11}dq^1dq^1 + g_{22}dq^2dq^2 + g_{33}dq^3dq^3,$$

hence

$$(ds)^2 = \frac{1}{2}dq^1dq^1 + \frac{1}{2}dq^2dq^2 + \frac{1}{4}dq^3dq^3.$$

(4) Consider $\mathbf{A} = 2\mathbf{i}_1$. As $A_i = \mathbf{A} \cdot \mathbf{e}_i$, we have

$$A_1 = 1, \quad A_2 = 1, \quad A_3 = 0.$$

Then using the relation $A^i = g^{ii}A_i$ (no summation), we obtain

$$A^1 = g^{11}A_1 = 2$$
$$A^2 = g^{22}A_2 = 2$$
$$A^3 = g^{33}A_3 = 0.$$

Similarly one has

$$B_1 = 1, \quad B_2 = 0, \quad B_3 = 0,$$
$$B^1 = 2, \quad B^2 = 0, \quad B^3 = 0$$

and

$$C_1 = 1, \quad C_2 = 1, \quad C_3 = -3/2,$$
$$C^1 = 2, \quad C^2 = 2, \quad C^3 = -6.$$

5.9 Given two generalised coordinate systems, with bases $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ and $\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3$, respectively, suppose that the position vector $\mathbf{x}$ has coordinates (x^1, x^2, x^3) with respect to $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ and suppose that $\mathbf{x}$ has coordinates (x'^1, x'^2, x'^3) with respect to $\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3$.

Let $L^j_{i'}$ be the coefficients of the direct transformation and $L^{j'}_i$ the coefficients of the inverse transformation. Given the following alternative definitions:

$$L^j_{i'} = \frac{\partial x^j}{\partial x'^i}, \quad L^{i'}_j = \frac{\partial x'^i}{\partial x^j},$$

show that

$$g_i^{\cdot j} = L^{k'}_i L^j_{k'}.$$

Solution: We have

$$L^{k'}_i L^j_{k'} = \frac{\partial x'^k}{\partial x^i}\frac{\partial x^j}{\partial x'^k} = \frac{\partial x^j}{\partial x^i} = \delta_i^{\cdot j},$$

as required (using the notation for generalised coordinate systems).
