Tensor Analysis – Practical 5

Information:

- Please make sure to complete **all** exercises **before** the next lecture.
- The exercises marked with [**See lecture**] were solved in class.
- The exercises are **not organised by difficulty**.

5.1 [**See Lecture**] Write down the transformation rule for:

(*1*) a tensor of rank four,
(*2*) a tensor of rank six.

5.2 If $Q_{ijk\ell}$ is a tensor of rank four, show that $Q_{ijj\ell}$ is a tensor of rank two.

5.3 [**See lecture**] If T_{ij} is a tensor, show that T_{ii} is a scalar.

5.4 Consider the $2D$ polar coordinates $(x^1, x^2) = (r, \theta)$. In this system, the position vector $\mathbf{r}$ is given by

$$\mathbf{r} = r\cos\theta\,\mathbf{i}_1 + r\sin\theta\,\mathbf{i}_2,$$

where $\mathbf{i}_1$, $\mathbf{i}_2$ are the usual 2-dimensional Cartesian basis vectors.

(*1*) Find the corresponding basis vectors $\mathbf{e}_1$ and $\mathbf{e}_2$.
(*2*) Find the covariant metric tensor g_{ij} and give your answer as a 2×2 matrix. Using the fact that the basis $\mathbf{e}_1, \mathbf{e}_2$ is orthogonal, find the contravariant metric tensor g^{ij}.
(*3*) Give an expression for the arc length of the polar coordinate system, plus explicitly state the metric coefficients h_1, h_2.

5.5 In spherical coordinates $(x^1, x^2, x^3) = (r, \phi, \theta)$, the position vector $\mathbf{r}$ is given by

$$\mathbf{r} = r\sin\phi\cos\theta\,\mathbf{i}_1 + r\sin\phi\sin\theta\,\mathbf{i}_2 + r\cos\phi\,\mathbf{i}_3.$$

Find the basis vectors and the metric coefficients.

5.6 [**See Lecture**] Given two vectors $\mathbf{A}$ and $\mathbf{B}$, with components A_i and B_i, show that the second-order tensor with components

$$C_{ik} = A_iB_k - A_kB_i$$

is antisymmetric.

5.7 [Question from the Main Exam 23-24] The third-rank tensor T_{ijk} is symmetric with respect to its last two suffixes but antisymmetric with respect to its first and second suffices. Show that all entries of this tensor are zero (i.e. $T_{ijk} = 0$, for all choices of i,j,k).

5.8 Let q^1, q^2, q^3 be coordinates related to orthogonal coordinates x_1, x_2, x_3 with orthonormal basis $\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3$ by the formulae

$$q^1 = x_1 + x_2, \quad q^2 = x_1 - x_2, \quad q^3 = 2x_3.$$

(*1*) Find the corresponding basis vectors and show that they form an orthogonal basis. [Hint: Using $\mathbf{r} = x_1\mathbf{i}_1 + x_2\mathbf{i}_2 + x_3\mathbf{i}_3$, you need to find expressions for x_1, x_2, x_3 in terms of q^1, q^2, q^3 and substitute these into your expansion for $\mathbf{r}$.]

(*2*) Find the metric tensor g_{ik} and give your answer as a matrix. Using the fact that the basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ is orthogonal, find the contravariant metric tensor g^{ik}.

(*3*) Give an expression for the arc length of the coordinate system with respect to the basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$.

(*4*) Find the covariant and contravariant components of the vectors

$$\mathbf{A} = 2\mathbf{i}_1, \quad \mathbf{B} = \mathbf{i}_1 + \mathbf{i}_2, \quad \mathbf{C} = 2\mathbf{i}_1 - 3\mathbf{i}_3,$$

with respect to the basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$, but without computing the dual basis $\mathbf{e}^1, \mathbf{e}^2, \mathbf{e}^3$.

5.9 Given two generalised coordinate systems, with bases $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ and $\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3$, respectively, suppose that the position vector $\mathbf{x}$ has coordinates (x^1, x^2, x^3) with respect to $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ and suppose that $\mathbf{x}$ has coordinates (x'^1, x'^2, x'^3) with respect to $\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3$.

Let $L^j_{i'}$ be the coefficients of the direct transformation and $L^{j'}_i$ the coefficients of the inverse transformation. Given the following alternative definitions:

$$L^j_{i'} = \frac{\partial x^j}{\partial x'^i}, \quad L^{i'}_j = \frac{\partial x'^i}{\partial x^j},$$

show that

$$g^{\cdot j}_i = L^{k'}_i L^j_{k'}.$$