Tensor Analysis – Practical 6

Solutions

Information:

- Please make sure to complete **all** exercises **before** the next lecture.
- The exercises marked with [**See lecture**] were solved in class.
- The exercises are **not organised by difficulty**.

6.1 Using the alternative definitions of $L^k_{i'}$ and $L^{i'}_k$ i.e.,

$$L^k_{i'} = \frac{\partial x^k}{\partial x'^i}, \quad L^{i'}_k = \frac{\partial x'^i}{\partial x^k},$$

show that

$$L^i_{k'} L^{k'}_j = \delta^i_{.j},$$

where here $\delta^i_{.j}$ is the usual Kronecker delta, but just written according to the notation in generalised coordinate systems.

Solution: We have

$$L^i_{k'} L^{k'}_j = \frac{\partial x^i}{\partial x'^k} \frac{\partial x'^k}{\partial x^j} = \frac{\partial x^i}{\partial x^j} = \delta^i_{.j},$$

as required.

6.2 [**See Lecture**] Let $A_{ik\ell}$ be a covariant tensor of order 3 and B^{pqmn} a contravariant tensor of order 4. Prove that $A_{ik\ell}B^{k\ell mn}$ is a mixed tensor of order 3 (with one covariant and two contravariant indices).

Solution: As $A_{ik\ell}$ is a covariant tensor and B^{pqmn} is a contravariant tensor, we have

$$A'_{ik\ell} = L^p_{i'} L^q_{k'} L^r_{\ell'} A_{pqr} \quad \text{and} \quad B'^{k\ell mn} = L^{k'}_s L^{\ell'}_t L^{m'}_u L^{n'}_v B^{stuv}.$$

Hence

$$
\begin{aligned}
(A_{ik\ell}B^{k\ell mn})' &= A'_{ik\ell}B'^{k\ell mn} \\
&= L^p_{i'}L^q_{k'}L^r_{\ell'}A_{pqr}L^{k'}_s L^{\ell'}_t L^{m'}_u L^{n'}_v B^{stuv} \\
&= L^p_{i'}(L^q_{k'}L^{k'}_s)(L^r_{\ell'}L^{\ell'}_t)L^{m'}_u L^{n'}_v A_{pqr}B^{stuv} \\
&= L^p_{i'}(\delta^q_{.s})(\delta^r_{.t})L^{m'}_u L^{n'}_v A_{pqr}B^{stuv} \quad \text{(using Question 1)} \\
&= L^p_{i'}L^{m'}_u L^{n'}_v A_{pqr}B^{qruv},
\end{aligned}
$$

which shows that $A_{ik\ell}B^{k\ell mn}$ is a mixed tensor with one covariant and two contravariant indices, as required.

6.3 Which of the following relations between associated tensors are correct? Explain.

(1) $T^p_q = g^r_p T_{rq}$,

(2) $S^{pq} = g^{rp}g_{sq}S_{rs}$,

(3) $W^{p}_{\cdot rs} = g_{sq}W^{pq}_{\cdot\cdot s}$,

(4) $V^{qm\cdot tk}_{\cdot\cdot n} = g_{pk}g^{sn}g_{rm}V^{q\cdot st}_{\cdot r\cdot\cdot p}$.

Solution: None of them. In **(1)**, a dot is missing in one suffix. This is necessary to specify the order of occurrence of indices. We also need to raise r and call it p. The correct expression is $T^{p}_{\cdot q} = g^{rp}T_{rq}$.

In **(2)**, we see that s is a dummy suffix on the same level (both on the bottom), which is not allowed in a generalised coordinate systems. Same for **(3)** and **(4)**. Notice that **(3)** also is missing an r on the right-hand side.

6.4 Given a Cartesian coordinate system K with orthonormal basis $\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3$, consider the second-order tensor with components

$$[P_{ik}] = [P^{ik}] = [P_i^{\cdot k}] = [P^i_{\cdot k}] = \begin{pmatrix} 1 & 1 & -1 \\ 2 & 3 & 0 \\ 0 & -2 & 1 \end{pmatrix}.$$

Let K' be a new coordinate system with basis vectors

$$\begin{aligned} \mathbf{e}_1 &= \mathbf{i}_1, \\ \mathbf{e}_2 &= \mathbf{i}_1 - \mathbf{i}_2, \\ \mathbf{e}_3 &= \mathbf{i}_1 + \mathbf{i}_2 + 2\mathbf{i}_3. \end{aligned}$$

(1) Compute the dual basis vectors $\mathbf{e}^1, \mathbf{e}^2, \mathbf{e}^3$.

(2) Using (1) where possible, express the covariant, contravariant and mixed components of the given tensor in the system K'.

Solution: **(1)** Using the dual basis formula (check previous practical sheets if needed), we obtain:

$$\mathbf{e}^1 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \quad \mathbf{e}^2 = \begin{pmatrix} 0 \\ -1 \\ 1/2 \end{pmatrix}, \quad \mathbf{e}^3 = \begin{pmatrix} 0 \\ 0 \\ 1/2 \end{pmatrix}.$$

(2) We have

$$P'_{ik} = L^\ell_{i'} L^m_{k'} P_{\ell m},$$

where $L^p_{q'} = \mathbf{e}_q \cdot \mathbf{i}_p$ are the coefficients of the direct transformation above, i.e.,

$$\begin{aligned} &L^1_{1'} = 1, \quad L^2_{1'} = 0, \quad L^3_{1'} = 0, \\ &L^1_{2'} = 1, \quad L^2_{2'} = -1, \quad L^3_{2'} = 0, \\ &L^1_{3'} = 1, \quad L^2_{3'} = 1, \quad L^3_{3'} = 2. \end{aligned}$$

For instance,

$$L^1_{3'} = \mathbf{e}_3 \cdot \mathbf{i}_1 = (\mathbf{i}_1 + \mathbf{i}_2 + 2\mathbf{i}_3) \cdot \mathbf{i}_1 = 1.$$

To compute $[P'_{ik}]$, we use the formula

$$P'_{ik} = L^\ell_{i'} L^m_{k'} P_{\ell m}.$$

If we had to use the definition, we would have for instance,

$$P'_{23} = L^\ell_{2'} L^m_{3'} P_{\ell m} = \sum_{\ell=1}^{3} \sum_{m=1}^{3} L^\ell_{2'} L^m_{3'} P_{\ell m},$$

which is a lot of work. So, instead, we can define the matrix $L = [L^i_{j'}]$ (and write $P = [P_{\ell m}]$) and notice that

$$(LP)_{im} = (L)_{i\ell}(P)_{\ell m} = L^\ell_{i'} P_{\ell m}.$$

whereas

$$(PL^T)_{\ell m} = (P)_{\ell m}(L^T)_{mk} = (P)_{\ell m}(L)_{km} = P_{\ell m}L^m_{k'}$$

Thus, putting all together, we get

$$(LPL^T)_{ij} = (L)_{i\ell}(PL^T)_{\ell m} = (L)_{i\ell}P_{\ell m}L^m_{k'} = (LP)_{im}L^m_{k'} = L^\ell_{i'}P_{\ell m}L^m_{k'}.$$

Hence, we only need to perform matrix multiplications:

$$\begin{aligned} P' = [P'_{ik}] = LPL^T &= \begin{pmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ 1 & 1 & 2 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 & -1 \\ 2 & 3 & 0 \\ 0 & -2 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ 1 & 1 & 2 \end{pmatrix}^T \\ &= \begin{pmatrix} 1 & 1 & -1 \\ -1 & -2 & -1 \\ 3 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & -5 \\ 3 & 3 & 5 \end{pmatrix}. \end{aligned}$$

To find P'^{ik}, $P'^{.k}_i$ and $P'^i_{.k}$, we use the formulae

$$\begin{aligned} P'^{ik} &= g^{i\ell}g^{km}P'_{\ell m}, \\ P'^{.k}_i &= g^{k\ell}P'_{i\ell}, \\ P'^i_{.k} &= g^{i\ell}P'_{\ell k}, \end{aligned}$$

Writing G for the matrix defined by g^{pq} and recalling that $g^{pq} = g^{qp}$, we rewrite the about suffix notation equations in matrix form:

$$\begin{aligned} P'^{ik} &= g^{i\ell}P'_{\ell m}g^{mk} = (GP'G)_{ik}, \\ P'^{.k}_i &= P'_{i\ell}g^{\ell k} = (P'G)_{ik}, \\ P'^i_{.k} &= g^{i\ell}P'_{\ell k} = (GP')_{ik}. \end{aligned}$$

First note that

$$G = [g^{ik}] = [\mathbf{e}^i \cdot \mathbf{e}^k] = \begin{pmatrix} 3 & -3/2 & -1/2 \\ -3/2 & 5/4 & 1/4 \\ -1/2 & 1/4 & 1/4 \end{pmatrix}.$$

As a result, we obtain

$$
\begin{aligned}
[P'^{ik}] &= \begin{pmatrix} 3 & -3/2 & -1/2 \\ -3/2 & 5/4 & 1/4 \\ -1/2 & 1/4 & 1/4 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & -5 \\ 3 & 3 & 5 \end{pmatrix} \begin{pmatrix} 3 & -3/2 & -1/2 \\ -3/2 & 5/4 & 1/4 \\ -1/2 & 1/4 & 1/4 \end{pmatrix} \\
&= \begin{pmatrix} 11 & -7 & -1 \\ -13/2 & 17/4 & 1/4 \\ -3/2 & 5/4 & 1/4 \end{pmatrix}, \\
[P'^{\cdot k}_{i}] &= \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & -5 \\ 3 & 3 & 5 \end{pmatrix} \begin{pmatrix} 3 & -3/2 & -1/2 \\ -3/2 & 5/4 & 1/4 \\ -1/2 & 1/4 & 1/4 \end{pmatrix} \\
&= \begin{pmatrix} 3 & -3/2 & -1/2 \\ -2 & 3/2 & -1/2 \\ 2 & 1/2 & 1/2 \end{pmatrix}, \\
[P'^{i}_{\cdot k}] &= \begin{pmatrix} 3 & -3/2 & -1/2 \\ -3/2 & 5/4 & 1/4 \\ -1/2 & 1/4 & 1/4 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & -5 \\ 3 & 3 & 5 \end{pmatrix} \\
&= \begin{pmatrix} 3 & -3 & 5 \\ -2 & 2 & -5 \\ 0 & 1 & 0 \end{pmatrix}.
\end{aligned}
$$

6.5 Given a Cartesian coordinate system K with orthonormal basis $\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3$, consider the second-order tensor with components

$$
[P_{ik}] = [P^{ik}] = [P_{i}^{\cdot k}] = [P^{i}_{\cdot k}] = \begin{pmatrix} 2 & 0 & 0 \\ 2 & 0 & -1 \\ 1 & 0 & 0 \end{pmatrix}.
$$

Let K' be a new coordinate system with basis vectors

$$
\begin{aligned}
\mathbf{e}_1 &= \mathbf{i}_1 - \mathbf{i}_2, \\
\mathbf{e}_2 &= \mathbf{i}_2, \\
\mathbf{e}_3 &= \mathbf{i}_1 + 2\mathbf{i}_3.
\end{aligned}
$$

(*1*) Compute the dual basis vectors $\mathbf{e}^1, \mathbf{e}^2, \mathbf{e}^3$.

(*2*) Using (1) where possible, express the covariant, contravariant and mixed components of the given tensor in the system K'.

Solution:

(1) Using the dual basis formula, we obtain:

$$\mathbf{e}^1 = \begin{pmatrix} 1 \\ 0 \\ -1/2 \end{pmatrix}, \quad \mathbf{e}^2 = \begin{pmatrix} 1 \\ 1 \\ -1/2 \end{pmatrix}, \quad \mathbf{e}^3 = \begin{pmatrix} 0 \\ 0 \\ 1/2 \end{pmatrix}.$$

(2) The matrix $L = [L^i_{j'}]$ is given by the identity $L^p_{q'} = \mathbf{e}_q \cdot \mathbf{i}_p$:

$$L = \begin{pmatrix} L^1_{1'} & L^2_{1'} & L^3_{1'} \\ L^1_{2'} & L^2_{2'} & L^3_{2'} \\ L^1_{3'} & L^2_{3'} & L^3_{3'} \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix}.$$

As in Question 6.4, to find $P' = [P'_{ik}]$, we only need to show that $P' = LPL^T$ and perform matrix multiplications

$$\begin{aligned} P' = [P'_{ik}] = LPL^T &= \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix} \cdot \begin{pmatrix} 2 & 0 & 0 \\ 2 & 0 & -1 \\ 1 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 1 \\ 2 & 0 & -1 \\ 4 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 2 \\ 2 & 0 & 0 \\ 4 & 0 & 4 \end{pmatrix}. \end{aligned}$$

To find P'^{ik}, $P'^{.k}_{i}$ and $P'^{i}_{.k}$, we use the formulae

$$\begin{aligned} P'^{ik} &= g^{i\ell} g^{km} P'_{\ell m}, \\ P'^{.k}_{i} &= g^{k\ell} P'_{i\ell}, \\ P'^{i}_{.k} &= g^{i\ell} P'_{\ell k}, \end{aligned}$$

Writing G for the matrix defined by g^{pq} and recalling that $g^{pq} = g^{qp}$, we rewrite the about suffix notation equations in matrix form:

$$\begin{aligned} P'^{ik} &= g^{i\ell} P'_{\ell m} g^{mk} = (GP'G)_{ik}, \\ P'^{.k}_{i} &= P'_{i\ell} g^{\ell k} = (P'G)_{ik}, \\ P'^{i}_{.k} &= g^{i\ell} P'_{\ell k} = (GP')_{ik}. \end{aligned}$$

First note that

$$G = [g^{ik}] = [\mathbf{e}^i \cdot \mathbf{e}^k] = \begin{pmatrix} 5/4 & 5/4 & -1/4 \\ 5/4 & 9/4 & -1/4 \\ -1/4 & -1/4 & 1/4 \end{pmatrix}.$$

As a result, we obtain

$$
\begin{aligned}
[P'^{ik}] &= \begin{pmatrix} 5/4 & 5/4 & -1/4 \\ 5/4 & 9/4 & -1/4 \\ -1/4 & -1/4 & 1/4 \end{pmatrix} \begin{pmatrix} 0 & 0 & 2 \\ 2 & 0 & 0 \\ 4 & 0 & 4 \end{pmatrix} \begin{pmatrix} 5/4 & 5/4 & -1/4 \\ 5/4 & 9/4 & -1/4 \\ -1/4 & -1/4 & 1/4 \end{pmatrix} \\
&= \begin{pmatrix} 3/2 & 0 & 3/2 \\ 7/2 & 0 & 3/2 \\ 1/2 & 0 & 1/2 \end{pmatrix} \cdot \begin{pmatrix} 5/4 & 5/4 & -1/4 \\ 5/4 & 9/4 & -1/4 \\ -1/4 & -1/4 & 1/4 \end{pmatrix} = \begin{pmatrix} 3/2 & 3/2 & 0 \\ 4 & 4 & -1/2 \\ 1/2 & 1/2 & 0 \end{pmatrix}, \\
[P'^{.k}_{i}] &= \begin{pmatrix} 0 & 0 & 2 \\ 2 & 0 & 0 \\ 4 & 0 & 4 \end{pmatrix} \begin{pmatrix} 5/4 & 5/4 & -1/4 \\ 5/4 & 9/4 & -1/4 \\ -1/4 & -1/4 & 1/4 \end{pmatrix} \\
&= \begin{pmatrix} -1/2 & -1/2 & 1/2 \\ 5/2 & -5/2 & -1/2 \\ 4 & 4 & 0 \end{pmatrix}, \\
[P'^{i}_{.k}] &= \begin{pmatrix} 5/4 & 5/4 & -1/4 \\ 5/4 & 9/4 & -1/4 \\ -1/4 & -1/4 & 1/4 \end{pmatrix} \begin{pmatrix} 0 & 0 & 2 \\ 2 & 0 & 0 \\ 4 & 0 & 4 \end{pmatrix} \\
&= \begin{pmatrix} 3/2 & 0 & 3/2 \\ 7/2 & 0 & 3/2 \\ 1/2 & 0 & 1/2 \end{pmatrix}.
\end{aligned}
$$

6.6 **[See lecture]** Given a Cartesian coordinate system K with orthonormal basis $\mathbf{i}_1$, $\mathbf{i}_2$, $\mathbf{i}_3$, consider the second-order tensor with components

$$[B_{ik}] = [B^{ik}] = [B_{i}^{.k}] = [B^{i}_{.k}] = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Let K' be a new coordinate system with basis vectors

$$
\begin{aligned}
\mathbf{e}_1 &= \mathbf{i}_1 - \mathbf{i}_2, \\
\mathbf{e}_2 &= \mathbf{i}_1 + 4\mathbf{i}_2 + \mathbf{i}_3, \\
\mathbf{e}_3 &= \mathbf{i}_1 + 3\mathbf{i}_2 + \mathbf{i}_3.
\end{aligned}
$$

(1) Compute the covariant components of B in the system K'.
(2) Compute the dual basis vectors $\mathbf{e}^1, \mathbf{e}^2, \mathbf{e}^3$.
(3) Using the metric tensor, compute contravariant components.

Solution: **(1)** We need to find the matrix $L = (L^j_{i'})$ satisfying

$$\begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix} = L \begin{pmatrix} \mathbf{i}_1 \\ \mathbf{i}_2 \\ \mathbf{i}_3 \end{pmatrix}.$$

We see that

$$L = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 4 & 1 \\ 1 & 3 & 1 \end{pmatrix}.$$

We know that the covariant components of $\mathbf{B}$ are given by $B'_{ik} = L^\ell_{i'} L^m_{k'} B_{\ell m} = [L\mathbf{B}L^T]_{ik}$, so we only need to compute $L\mathbf{B}L^T$:

$$\begin{aligned} \mathbf{B}' = L\mathbf{B}L^T &= \begin{pmatrix} 1 & -1 & 0 \\ 1 & 4 & 1 \\ 1 & 3 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 \\ 1 & 4 & 1 \\ 1 & 3 & 1 \end{pmatrix}^T \\ &= \begin{pmatrix} 1 & 1 & -1 \\ -4 & 1 & 5 \\ -3 & 1 & 4 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ -1 & 4 & 3 \\ 0 & 1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 4 & 3 \\ -5 & 5 & 4 \\ -4 & 5 & 4 \end{pmatrix} \end{aligned}$$

(2) Using the dual basis formula, we obtain:

$$\mathbf{e}^1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \quad \mathbf{e}^2 = \begin{pmatrix} 1 \\ 1 \\ -4 \end{pmatrix}, \quad \mathbf{e}^3 = \begin{pmatrix} -1 \\ -1 \\ 5 \end{pmatrix}.$$

(3) To find B'^{ik}, $B'^{\cdot k}_i$ and $B'^i_{\cdot k}$, we use the formulae

$$\begin{aligned} B'^{ik} &= g^{i\ell} g^{km} B'_{\ell m}, \\ B'^{\cdot k}_i &= g^{k\ell} B'_{i\ell}, \\ B'^i_{\cdot k} &= g^{i\ell} B'_{\ell k}, \end{aligned}$$

Writing G for the matrix defined by g^{pq} and recalling that $g^{pq} = g^{qp}$, we rewrite the about suffix notation equations in matrix form:

$$\begin{aligned} B'^{ik} &= g^{i\ell} B'_{\ell m} g^{mk} = (G\mathbf{B}'G)_{ik}, \\ B'^{\cdot k}_i &= B'_{i\ell} g^{\ell k} = (\mathbf{B}'G)_{ik}, \\ B'^i_{\cdot k} &= g^{i\ell} B'_{\ell k} = (G\mathbf{B}')_{ik}. \end{aligned}$$

First note that

$$G = [g^{ik}] = [\mathbf{e}^i \cdot \mathbf{e}^k] = \begin{pmatrix} 2 & 5 & -6 \\ 5 & 18 & -22 \\ -6 & -22 & 27 \end{pmatrix}.$$

As a result, we obtain

$$\begin{aligned}
[B'^{ik}] = G\,\mathbf{B}'\,G &= \begin{pmatrix} 2 & 5 & -6 \\ 5 & 18 & -22 \\ -6 & -22 & 27 \end{pmatrix} \begin{pmatrix} 0 & 4 & 3 \\ -5 & 5 & 4 \\ -4 & 5 & 4 \end{pmatrix} \begin{pmatrix} 2 & 5 & -6 \\ 5 & 18 & -22 \\ -6 & -22 & 27 \end{pmatrix} \\
&= \begin{pmatrix} -1 & 3 & 2 \\ -2 & 0 & 1 \\ 2 & 1 & 2 \end{pmatrix} \cdot \begin{pmatrix} 2 & 5 & -6 \\ 5 & 18 & -22 \\ -6 & -22 & 27 \end{pmatrix} = \begin{pmatrix} 1 & 5 & -6 \\ 2 & 12 & -15 \\ -3 & -16 & 20 \end{pmatrix}, \\
[B'^{.k}_{i}] = \mathbf{B}'G &= \begin{pmatrix} 0 & 4 & 3 \\ -5 & 5 & 4 \\ -4 & 5 & 4 \end{pmatrix} \begin{pmatrix} 2 & 5 & -6 \\ 5 & 18 & -22 \\ -6 & -22 & 27 \end{pmatrix} \\
&= \begin{pmatrix} 2 & 6 & -7 \\ -9 & -23 & 28 \\ -7 & -18 & 22 \end{pmatrix}, \\
[B'^{i}_{.k}] = G\,\mathbf{B}' &= \begin{pmatrix} 2 & 5 & -6 \\ 5 & 18 & -22 \\ -6 & -22 & 27 \end{pmatrix} \begin{pmatrix} 0 & 4 & 3 \\ -5 & 5 & 4 \\ -4 & 5 & 4 \end{pmatrix} \\
&= \begin{pmatrix} -1 & 3 & 2 \\ -2 & 0 & -1 \\ 2 & 1 & 2 \end{pmatrix}.
\end{aligned}$$
