

Tensor Analysis – Practical 6

Information:

- Please make sure to complete **all** exercises **before** the next lecture.
- The exercises marked with **[See lecture]** were solved in class.
- The exercises are **not organised by difficulty**.

6.1 Using the alternative definitions of $L_{i'}^k$ and $L_k^{i'}$ i.e.,

$$L_{i'}^k = \frac{\partial x^k}{\partial x^{i'}}, \quad L_k^{i'} = \frac{\partial x^{i'}}{\partial x^k},$$

show that

$$L_{k'}^i L_j^{k'} = \delta_{.j}^i,$$

where here $\delta_{.j}^i$ is the usual Kronecker delta, but just written according to the notation in generalised coordinate systems.

6.2 [See Lecture] Let $A_{ik\ell}$ be a covariant tensor of order 3 and B^{pqmn} a contravariant tensor of order 4. Prove that $A_{ik\ell} B^{k\ell mn}$ is a mixed tensor of order 3 (with one covariant and two contravariant indices).

6.3 Which of the following relations between associated tensors are correct? Explain.

- (1) $T_q^p = g_p^r T_{rq}$,
- (2) $S^{pq} = g^{rp} g_{sq} S_{rs}$,
- (3) $W_{.rs}^p = g_{sq} W_{.s}^{pq}$,
- (4) $V_{.n}^{qm.tk} = g_{pk} g^{sn} g_{rm} V_{.r.p}^{q.st}$.

6.4 Given a Cartesian coordinate system K with orthonormal basis $\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3$, consider the second-order tensor with components

$$[P_{ik}] = [P^{ik}] = [P_i^k] = [P^i_k] = \begin{pmatrix} 1 & 1 & -1 \\ 2 & 3 & 0 \\ 0 & -2 & 1 \end{pmatrix}.$$

Let K' be a new coordinate system with basis vectors

$$\begin{aligned} \mathbf{e}_1 &= \mathbf{i}_1, \\ \mathbf{e}_2 &= \mathbf{i}_1 - \mathbf{i}_2, \\ \mathbf{e}_3 &= \mathbf{i}_1 + \mathbf{i}_2 + 2\mathbf{i}_3. \end{aligned}$$

- (1) Compute the dual basis vectors $\mathbf{e}^1, \mathbf{e}^2, \mathbf{e}^3$.

- (2) Using (1) where possible, express the covariant, contravariant and mixed components of the given tensor in the system K' .

6.5 Given a Cartesian coordinate system K with orthonormal basis $\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3$, consider the second-order tensor with components

$$[P_{ik}] = [P^{ik}] = [P_i^{\cdot k}] = [P^i_{\cdot k}] = \begin{pmatrix} 2 & 0 & 0 \\ 2 & 0 & -1 \\ 1 & 0 & 0 \end{pmatrix}.$$

Let K' be a new coordinate system with basis vectors

$$\mathbf{e}_1 = \mathbf{i}_1 - \mathbf{i}_2,$$

$$\mathbf{e}_2 = \mathbf{i}_2,$$

$$\mathbf{e}_3 = \mathbf{i}_1 + 2\mathbf{i}_3.$$

- (1) Compute the dual basis vectors $\mathbf{e}^1, \mathbf{e}^2, \mathbf{e}^3$.
 (2) Using (1) where possible, express the covariant, contravariant and mixed components of the given tensor in the system K' .

6.6 [See lecture] Given a Cartesian coordinate system K with orthonormal basis $\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3$, consider the second-order tensor with components

$$[B_{ik}] = [B^{ik}] = [B_i^{\cdot k}] = [B^i_{\cdot k}] = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Let K' be a new coordinate system with basis vectors

$$\mathbf{e}_1 = \mathbf{i}_1 - \mathbf{i}_2,$$

$$\mathbf{e}_2 = \mathbf{i}_1 + 4\mathbf{i}_2 + \mathbf{i}_3,$$

$$\mathbf{e}_3 = \mathbf{i}_1 + 3\mathbf{i}_2 + \mathbf{i}_3.$$

- (1) Compute the covariant components of B in the system K' .
 (2) Compute the dual basis vectors $\mathbf{e}^1, \mathbf{e}^2, \mathbf{e}^3$.
 (3) Using the metric tensor, compute contravariant components.