Tensor Analysis – Practical 7

Solutions

Information:

- Please make sure to complete **all** exercises **before** the next lecture.
- The exercises marked with [**See lecture**] were solved in class.
- The exercises are **not organised by difficulty**.

7.1 Compute the outer product following pairs of tensors.

(1) $M = [M_{ij}] = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$ and $N = [N_{ij}] = \begin{pmatrix} 0 & 2 \\ 1 & 3 \end{pmatrix}$.

(2) $P = [P_i] = \begin{pmatrix} 0 & 1 \end{pmatrix}$ and

$$Q = [Q_{ijkl}] = \begin{pmatrix} Q_{1111} & Q_{1112} & Q_{1211} & Q_{1212} \\ Q_{1121} & Q_{1122} & Q_{1221} & Q_{1222} \\ Q_{2111} & Q_{2112} & Q_{2211} & Q_{2212} \\ Q_{2121} & Q_{2122} & Q_{2221} & Q_{2222} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 2 & 0 \\ 1 & 1 & -1 & -1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & -1 & 3 \end{pmatrix}.$$

(3) $R = [R_i] = \begin{pmatrix} 1 & -1 & 0 \end{pmatrix}$ and $S = [S_{ij}] = \begin{pmatrix} 1 & 2 & 4 \\ -1 & 0 & -7 \\ 0 & 1 & 0 \end{pmatrix}$.

Solution: (1) By definition

$$M \otimes N = \begin{pmatrix} 0N & 1N \\ 1N & -1N \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 2 & 0 & -2 \\ 1 & 3 & -1 & -3 \end{pmatrix}$$

(2) By definition

$$P \otimes Q = \begin{pmatrix} 0Q & 1Q \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 & 3 \end{pmatrix}$$

(3) By definition

$$R \otimes S = \begin{pmatrix} 1S & -S & 0S \end{pmatrix} = \begin{pmatrix} 1 & 2 & 4 & -1 & -2 & -4 & 0 & 0 & 0 \\ -1 & 0 & -7 & 1 & 0 & 7 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

7.2 Let T_{ij} and $R_{ijk\ell}$ be tensors of rank 2 and 4, respectively. Show that

$$T_{km}R_{kpqm}$$

is a rank 2 tensor.

Solution: We must show

$$T_{km}R_{kpqm} = L_{pv}L_{qw}T_{n\ell}R_{nvw\ell}.$$

Because T_{ij} and $R_{ijk\ell}$ are tensors, we know that

$$\begin{aligned} T_{km} &= L_{kn}L_{m\ell}T_{n\ell}, \\ R_{kpqm} &= L_{ku}L_{pv}L_{qw}L_{ms}R_{uvws}. \end{aligned}$$

Thus,

$$\begin{aligned} T_{km}R_{kpqm} &= L_{kn}L_{m\ell}T_{n\ell}L_{ku}L_{pv}L_{qw}L_{ms}R_{uvws} \\ &= (L_{kn}L_{ku})\,(L_{m\ell}L_{ms})\,L_{pv}L_{qw}T_{n\ell}R_{uvws} && \text{(re-ordering)} \\ &= \delta_{nu}\delta_{\ell s}L_{pv}L_{qw}T_{n\ell}R_{uvws} && \text{(using } L_{ab}L_{ac} = \delta_{bc}\text{)} \\ &= L_{pv}L_{qw}T_{n\ell}\,(\delta_{nu}\delta_{\ell s}R_{uvws}) && \text{(re-ordering)} \\ &= L_{pv}L_{qw}T_{n\ell}R_{nvw\ell} \end{aligned}$$

7.3 Given that

$$[R_{ik}] = \begin{pmatrix} 1 & 0 & 2 \\ 3 & 2 & 1 \\ 1 & 3 & 4 \end{pmatrix}, \qquad \mathbf{A} = \mathbf{i}_1 + 2\mathbf{i}_2 - \mathbf{i}_3.$$

find the inner products $R_{ik}A_i$ and $R_{ik}A_k$.

Solution: Notice that $R_{ik}A_i$ can be rewritten as $R_{ik}A_i = (R^T)_{ki}A_i = (R^T A)_k$.

$$\begin{pmatrix} 1 & 0 & 2 \\ 3 & 2 & 1 \\ 1 & 3 & 4 \end{pmatrix}^T \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix},$$

which equals

$$\begin{pmatrix} 1 & 3 & 1 \\ 0 & 2 & 3 \\ 2 & 1 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 6 \\ 1 \\ 0 \end{pmatrix}.$$

Next, $T_{ik}A_k$ is the usual

$$\begin{pmatrix} 1 & 0 & 2 \\ 3 & 2 & 1 \\ 1 & 3 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix},$$

which equals

$$\begin{pmatrix} -1 \\ 6 \\ 3 \end{pmatrix}.$$

7.4 Let T_{ik} and $\mathbf{A}$ be the same as in the preceding problem, and let

$$\mathbf{B} = 4\mathbf{i}_1 + 5\mathbf{i}_2 + 3\mathbf{i}_3.$$

Find the inner product $T_{ik}A_iB_k$.

Solution:

We set $C_k := T_{ik}A_i$, and thus $C = \begin{pmatrix} 6 \\ 1 \\ 0 \end{pmatrix}$. Then $T_{ik}A_iB_k = C_kB_k$ which is the dot product of C with B. This equals $24 + 5 = 29$.

7.5 Find all possible contractions for the following dimension-two tensors.

(1) $[A_{ij}] = \begin{pmatrix} 1 & -1 \\ 4 & 0 \end{pmatrix}.$

(2)

$$[B_{ijk\ell}] = \begin{pmatrix} B_{1111} & B_{1112} & B_{1211} & B_{1212} \\ B_{1121} & B_{1122} & B_{1221} & B_{1222} \\ B_{2111} & B_{2112} & B_{2211} & B_{2212} \\ B_{2121} & B_{2122} & B_{2221} & B_{2222} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 7 \\ 4 & 0 & 2 & 11 \\ 2 & 2 & -1 & -3 \\ 1 & 128 & 2 & 0 \end{pmatrix}.$$

Solution:

(1) $A_{ii} = A_{11} + A_{22} = 1 + 0 = 1.$

(2) Here there are different options:

$$B_{iikk} = \sum_{i,k=1}^{2} B_{iikk} = \sum_{k=1}^{2} (B_{11kk} + B_{22kk}) = (B_{1111} + B_{2211}) + (B_{1122} + B_{2222}) = 1 - 1 + 0 + 0 = 0.$$

$$B_{ijij} = \sum_{i,j=1}^{2} B_{ijij} = \sum_{j=1}^{2} (B_{1j1j} + B_{2j2j}) = (B_{1111} + B_{2121}) + (B_{1212} + B_{2222}) = 1 + 1 + 7 + 0 = 9.$$

$$B_{ijji} = \sum_{i,j=1}^{2} B_{ijji} = \sum_{j=1}^{2} (B_{1jj1} + B_{2jj2}) = (B_{1111} + B_{2112}) + (B_{1221} + B_{2222}) = 1 + 2 + 2 + 0 = 5.$$
