

Tensor Analysis – Practical 8

Solutions

Information:

- Please make sure to complete **all** exercises **before** the next lecture.
- The exercises marked with **[See lecture]** were solved in class.
- The exercises are **not organised by difficulty**.

8.1 The Christoffel symbols of the first kind can be expressed in terms of the metric tensor as follows:

$$\Gamma_{ijk} = \frac{1}{2} \left(\frac{\partial g_{ik}}{\partial x^j} + \frac{\partial g_{ij}}{\partial x^k} - \frac{\partial g_{kj}}{\partial x^i} \right).$$

This expression simplifies for orthogonal coordinate systems (those with $g_{ij} = 0$ when $i \neq j$). Find the simplified expression for Γ_{ijk} in the cases

- (1) $i = j = k$;
- (2) $i = j \neq k$;
- (3) $i \neq j = k$;
- (4) i, j, k are all distinct.

Solution: (1) If $i = j = k$, then for fixed i (no summation convention),

$$\Gamma_{ijk} = \Gamma_{iii} = \frac{1}{2} \left(\frac{\partial g_{ii}}{\partial x^i} + \frac{\partial g_{ii}}{\partial x^i} - \frac{\partial g_{ii}}{\partial x^i} \right) = \frac{1}{2} \frac{\partial g_{ii}}{\partial x^i}.$$

(2) If $i = j \neq k$, then likewise

$$\Gamma_{ijk} = \Gamma_{iik} = \frac{1}{2} \left(\frac{\partial g_{ii}}{\partial x^k} + \frac{\partial g_{ik}}{\partial x^i} - \frac{\partial g_{ik}}{\partial x^i} \right) = \frac{1}{2} \frac{\partial g_{ii}}{\partial x^k}.$$

(3) If $i \neq j = k$ then

$$\Gamma_{ijk} = \Gamma_{ijj} = \frac{1}{2} \left(\frac{\partial g_{ij}}{\partial x^j} + \frac{\partial g_{ij}}{\partial x^j} - \frac{\partial g_{jj}}{\partial x^i} \right) = -\frac{1}{2} \frac{\partial g_{jj}}{\partial x^i},$$

as $g_{ij} = 0$ for $i \neq j$.

(d) If i, j, k are all distinct, then

$$\Gamma_{ijk} = \frac{1}{2} \left(\frac{\partial g_{ij}}{\partial x^k} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^i} \right) = 0.$$

8.2 [See lecture] We have seen in Practical 5 spherical coordinates $(x^1, x^2, x^3) = (r, \phi, \theta)$, the metric tensors are

$$g_{11} = 1, \quad g_{22} = r^2, \quad g_{33} = r^2 \sin^2 \phi,$$

and the other components of the metric tensor are trivial. Find the Christoffel symbols of the first and second kind.

Solution: We have a formula for Γ_{ijk} in terms of the metric tensors:

$$\Gamma_{ijk} = \frac{1}{2} \left(\frac{\partial g_{ij}}{\partial x^k} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^i} \right).$$

As the only non-constant g_{ij} are $g_{22} = r^2$ and $g_{33} = r^2 \sin^2 \phi$, the only non-zero partial derivatives $\frac{\partial g_{ij}}{\partial x^k}$ are

$$\begin{aligned} \frac{\partial g_{22}}{\partial x^1} &= \frac{\partial r^2}{\partial r} = 2r, \\ \frac{\partial g_{33}}{\partial x^1} &= \frac{\partial (r^2 \sin^2 \phi)}{\partial r} = 2r \sin^2 \phi, \\ \frac{\partial g_{33}}{\partial x^2} &= \frac{\partial (r^2 \sin^2 \phi)}{\partial \phi} = 2r^2 \sin \phi \cos \phi. \end{aligned}$$

So, we conclude that the only possibly non-zero Christoffel symbols are the ones with indices

- 1,2,2, or
- 1,3,3, or
- 2,3,3,

in some order. Thus

$$\Gamma_{122} = \frac{1}{2} \left(\frac{\partial g_{12}}{\partial x^2} + \frac{\partial g_{12}}{\partial x^2} - \frac{\partial g_{22}}{\partial x^1} \right) = -\frac{1}{2} \frac{\partial g_{22}}{\partial x^1} = -\frac{1}{2} \cdot 2r = -r,$$

$$\Gamma_{212} = \Gamma_{221} = \frac{1}{2} \frac{\partial g_{22}}{\partial x^1} = \frac{1}{2} \cdot 2r = r,$$

$$\Gamma_{133} = -\frac{1}{2} \frac{\partial g_{33}}{\partial x^1} = -\frac{1}{2} \cdot 2r \sin^2 \phi = -r \sin^2 \phi,$$

$$\Gamma_{313} = \Gamma_{331} = \frac{1}{2} \frac{\partial g_{33}}{\partial x^1} = \frac{1}{2} \frac{\partial g_{33}}{\partial x^1} = \frac{1}{2} \cdot 2r \sin^2 \phi = r \sin^2 \phi,$$

$$\Gamma_{233} = -\frac{1}{2} \frac{\partial g_{33}}{\partial x^2} = -\frac{1}{2} \cdot (2r^2 \sin \phi \cos \phi) = -r^2 \sin \phi \cos \phi,$$

$$\Gamma_{323} = \Gamma_{332} = \frac{1}{2} \frac{\partial g_{33}}{\partial x^2} = \frac{1}{2} \cdot (2r^2 \sin \phi \cos \phi) = r^2 \sin \phi \cos \phi$$

$$\Gamma_{ijk} = 0, \quad \text{for all other values of } i, j, k.$$

To find the Christoffel symbols of second kind Γ^i_{jk} we use the formula

$$\Gamma^i_{jk} = g^{i\ell} \Gamma_{\ell kj} = g^{i1} \Gamma_{1kj} + g^{i2} \Gamma_{2kj} + g^{i3} \Gamma_{3kj}.$$

Recall that in orthogonal coordinate systems $g^{ii} = \frac{1}{g_{ii}}$ and $g^{ij} = 0$ if $i \neq j$. Thus

$$g^{11} = 1, \quad g^{22} = \frac{1}{r^2}, \quad g^{33} = \frac{1}{r^2 \sin^2 \phi}.$$

Applying to the formula, we see that

$$\Gamma_{22}^1 = g^{11} \Gamma_{122} = -r,$$

$$\Gamma_{12}^2 = \Gamma_{21}^2 = g^{22} \Gamma_{212} = \frac{1}{r^2} \cdot r = \frac{1}{r},$$

$$\Gamma_{33}^1 = g^{11} \Gamma_{133} = -r \sin^2 \phi,$$

$$\Gamma_{13}^3 = \Gamma_{31}^3 = g^{33} \Gamma_{313} = \frac{1}{r^2 \sin^2 \phi} \cdot (r \sin^2 \phi) = \frac{1}{r},$$

$$\Gamma_{33}^2 = g^{22} \Gamma_{233} = \frac{1}{r^2} \cdot (-r^2 \sin \phi \cos \phi) = -\sin \phi \cos \phi,$$

$$\Gamma_{23}^3 = \Gamma_{32}^3 = g^{33} \Gamma_{323} = \frac{1}{r^2 \sin^2 \phi} \cdot (r^2 \sin \phi \cos \phi) = \cot \phi$$

$$\Gamma_{ijk} = 0, \quad \text{for all other values of } i, j, k.$$

8.3 Consider the **2D** coordinates $(x^1, x^2) = (\rho, \theta)$, in which the position vector \mathbf{r} is given by

$$\mathbf{r} = \rho^2 \cos(2\theta) \mathbf{i}_1 + \rho^2 \sin(2\theta) \mathbf{i}_2,$$

where $\mathbf{i}_1, \mathbf{i}_2$ are the usual 2-dimensional Cartesian basis vectors.

- (1) Find the corresponding basis vectors \mathbf{e}_1 and \mathbf{e}_2 .
- (2) Find the covariant metric tensor g_{ij} and give your answer as a 2×2 matrix. Using the fact that the basis $\mathbf{e}_1, \mathbf{e}_2$ is orthogonal, find the contravariant metric tensor g^{ij} .
- (3) Find all Christoffel symbols Γ_{ijk} of the first kind.
- (4) Find all Christoffel symbols Γ^i_{jk} of the second kind.

Solution: (1) We know that $e_i = \frac{\partial \mathbf{r}}{\partial x^i}$. Thus

$$\begin{aligned} e_1 &= \frac{\partial \mathbf{r}}{\partial x^1} = \frac{\partial \mathbf{r}}{\partial \rho} = 2\rho \cos(2\theta) \mathbf{i}_1 + 2\rho \sin(2\theta) \mathbf{i}_2, \\ e_2 &= \frac{\partial \mathbf{r}}{\partial x^2} = \frac{\partial \mathbf{r}}{\partial \theta} = -2\rho^2 \sin(2\theta) \mathbf{i}_1 + 2\rho^2 \cos(2\theta) \mathbf{i}_2. \end{aligned}$$

(2) By definition, $g_{ij} = \mathbf{e}_i \cdot \mathbf{e}_j$. Thus

$$\begin{aligned} g_{11} &= \begin{pmatrix} 2\rho \cos(2\theta) \\ 2\rho \sin(2\theta) \end{pmatrix} \cdot \begin{pmatrix} 2\rho \cos(2\theta) \\ 2\rho \sin(2\theta) \end{pmatrix} = 4\rho^2 \cos^2(2\theta) + 4\rho^2 \sin^2(2\theta) = 4\rho^2, \\ g_{12} = g_{21} &= \begin{pmatrix} 2\rho \cos(2\theta) \\ 2\rho \sin(2\theta) \end{pmatrix} \cdot \begin{pmatrix} -2\rho^2 \sin(2\theta) \\ 2\rho^2 \cos(2\theta) \end{pmatrix} = -4\rho^3 \cos(2\theta) \sin(2\theta) + 4\rho^3 \cos(2\theta) \sin(2\theta) = 0, \\ g_{22} &= \begin{pmatrix} -2\rho^2 \sin(2\theta) \\ 2\rho^2 \cos(2\theta) \end{pmatrix} \cdot \begin{pmatrix} -2\rho^2 \sin(2\theta) \\ 2\rho^2 \cos(2\theta) \end{pmatrix} = 4\rho^4 \sin^2(2\theta) + 4\rho^4 \cos^2(2\theta) = 4\rho^4. \end{aligned}$$

Note that since $g_{ij} = 0$ when $i \neq j$, this system is orthogonal. Thus, we can use the formula

$$g^{ii} = \frac{1}{g_{ii}}.$$

We obtain

$$g^{11} = \frac{1}{4\rho^2}, \quad g^{22} = \frac{1}{4\rho^4}.$$

By definition of orthogonal, we have $g^{12} = g^{21} = 0$.

In matrix notation, we have

$$G = [g_{ij}] = \begin{pmatrix} 4\rho^2 & 0 \\ 0 & 4\rho^4 \end{pmatrix}, \quad [g^{ij}] = \begin{pmatrix} \frac{1}{4\rho^2} & 0 \\ 0 & \frac{1}{4\rho^4} \end{pmatrix}.$$

(3) Recall that

$$(1) \quad \Gamma_{ijk} = \frac{1}{2} \left(\frac{\partial g_{ij}}{\partial x^k} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^i} \right).$$

Since

$$G = [g_{ij}] = \begin{pmatrix} 4\rho^2 & 0 \\ 0 & 4\rho^4 \end{pmatrix},$$

the only non-zero partial derivatives $\frac{\partial g_{ij}}{\partial x^k}$ are

$$\begin{aligned}\frac{\partial g_{11}}{\partial x^1} &= \frac{\partial(4\rho^2)}{\partial \rho} = 8\rho, \\ \frac{\partial g_{22}}{\partial x^1} &= \frac{\partial(4\rho^4)}{\partial \rho} = 16\rho^3.\end{aligned}$$

Thus, a quick analysis of the formula (1) reveals that the only possibly non-zero Christoffel symbols (of both first **and** second kind) are the ones with indices

- 1,1,1, or
- 1,1,2 (in some order).

Thus, the Christoffel symbols of first kind are given by

$$\begin{aligned}\Gamma_{111} &= \frac{1}{2} \left(\frac{\partial g_{11}}{\partial x^1} + \frac{\partial g_{11}}{\partial x^1} - \frac{\partial g_{11}}{\partial x^1} \right) = \frac{1}{2} \frac{\partial g_{11}}{\partial x^1} = \frac{1}{2} \cdot 8\rho = 4\rho, \\ \Gamma_{112} = \Gamma_{121} &= \frac{1}{2} \left(\frac{\partial g_{11}}{\partial x^2} + \frac{\partial g_{12}}{\partial x^1} - \frac{\partial g_{12}}{\partial x^1} \right) = \frac{1}{2} \frac{\partial g_{11}}{\partial x^2} = \frac{1}{2} \cdot 16\rho^3 = 8\rho^3, \\ \Gamma_{211} &= \frac{1}{2} \left(\frac{\partial g_{21}}{\partial x^1} + \frac{\partial g_{21}}{\partial x^1} - \frac{\partial g_{11}}{\partial x^2} \right) = -\frac{1}{2} \frac{\partial g_{11}}{\partial x^2} = \frac{1}{2} \cdot 16\rho^3 = -8\rho^3 \\ \Gamma_{ikj} &= 0, \quad \text{for all other values of } i, j, k.\end{aligned}$$

(4) To find the Christoffel symbols of second kind Γ^i_{jk} we use the formula

$$\Gamma^i_{jk} = g^{i\ell} \Gamma_{\ell kj} = g^{i1} \Gamma_{1kj} + g^{i2} \Gamma_{2kj}.$$

Since

$$\begin{aligned}[g^{ij}] &= \begin{pmatrix} \frac{1}{4\rho^2} & 0 \\ 0 & \frac{1}{4\rho^4} \end{pmatrix}. \\ \Gamma^1_{11} = g^{11} \Gamma_{111} &= \frac{1}{4\rho^2} \cdot 4\rho = \frac{1}{\rho}, \\ \Gamma^1_{12} = \Gamma^1_{21} = g^{11} \Gamma_{112} &= \frac{1}{4\rho^2} \cdot 8\rho^3 = 2\rho, \\ \Gamma^2_{11} = g^{22} \Gamma_{211} &= \frac{1}{4\rho^4} \cdot (-8\rho^3) = -\frac{2}{\rho}, \\ \Gamma^i_{kj} &= 0, \quad \text{for all other values of } i, j, k.\end{aligned}$$

8.4 Consider the coordinates $(x^1, x^2, x^3) = (x, y, t)$, in which the position vector \mathbf{r} is given by

$$\mathbf{r} = (x^2 - y^2) \mathbf{i}_1 + (2xy) \mathbf{i}_2 + t \mathbf{i}_3,$$

where $\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3$ are the usual Cartesian basis vectors.

- (1) Find the corresponding basis vectors $\mathbf{e}_1, \mathbf{e}_2$, and \mathbf{e}_3 .
- (2) Find the covariant metric tensor g_{ij} and the contravariant metric tensor g^{ij} . Give your final answer as 2×2 matrices.
- (3) Find all Christoffel symbols Γ_{ijk} of the first kind.
- (4) Find all Christoffel symbols Γ_{jk}^i of the second kind.

Solution: (1) We know that $e_i = \frac{\partial \mathbf{r}}{\partial x^i}$. Thus

$$\begin{aligned} e_1 &= \frac{\partial \mathbf{r}}{\partial x^1} = \frac{\partial \mathbf{r}}{\partial x} = 2x \mathbf{i}_1 + 2y \mathbf{i}_2, \\ e_2 &= \frac{\partial \mathbf{r}}{\partial x^2} = \frac{\partial \mathbf{r}}{\partial y} = -2y \mathbf{i}_1 + 2x \mathbf{i}_2 \\ e_3 &= \frac{\partial \mathbf{r}}{\partial x^3} = \frac{\partial \mathbf{r}}{\partial t} = \mathbf{i}_3. \end{aligned}$$

(2) By definition, $g_{ij} = \mathbf{e}_i \cdot \mathbf{e}_j$. Thus

$$\begin{aligned} g_{11} &= \begin{pmatrix} 2x \\ 2y \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 2x \\ 2y \\ 0 \end{pmatrix} = 4(x^2 + y^2), \\ g_{12} = g_{21} &= \begin{pmatrix} 2x \\ 2y \\ 0 \end{pmatrix} \cdot \begin{pmatrix} -2y \\ 2x \\ 0 \end{pmatrix} = -4xy + 4xy = 0, \\ g_{13} = g_{31} &= \begin{pmatrix} 2x \\ 2y \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 0, \end{aligned}$$

$$g_{22} = \begin{pmatrix} -2y \\ 2x \\ 0 \end{pmatrix} \cdot \begin{pmatrix} -2y \\ 2x \\ 0 \end{pmatrix} = 4(x^2 + y^2),$$

$$g_{23} = g_{32} = \begin{pmatrix} -2y \\ 2x \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 0,$$

$$g_{33} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 1.$$

Note that since $g_{ij} = 0$ when $i \neq j$, this system is **orthogonal**. Thus, we can use the formula

$$g^{ii} = \frac{1}{g_{ii}}.$$

We obtain

$$g^{11} = \frac{1}{4(x^2 + y^2)}, \quad g^{22} = \frac{1}{4(x^2 + y^2)}, \quad g^{33} = 1.$$

By definition of orthogonal, we also have $g^{ij} = 0$, whenever $i \neq j$.

Our final solution as matrices is then

$$G = [g_{ij}] = \begin{pmatrix} 4(x^2 + y^2) & 0 & 0 \\ 0 & 4(x^2 + y^2) & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad [g^{ij}] = \begin{pmatrix} \frac{1}{4(x^2 + y^2)} & 0 & 0 \\ 0 & \frac{1}{4(x^2 + y^2)} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

(3) To compute the Christoffel symbols of first kind, we use the following formula:

$$\Gamma_{ijk} = \frac{1}{2} \left(\frac{\partial g_{ij}}{\partial x^k} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^i} \right).$$

Since

$$G = [g_{ij}] = \begin{pmatrix} 4(x^2 + y^2) & 0 & 0 \\ 0 & 4(x^2 + y^2) & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

the only non-zero partial derivatives $\frac{\partial g_{ij}}{\partial x^k}$ are

$$\begin{aligned} \frac{\partial g_{11}}{\partial x^1} &= \frac{\partial g_{22}}{\partial x^1} = \frac{\partial(4(x^2 + y^2))}{\partial x} = 8x, \\ \frac{\partial g_{11}}{\partial x^2} &= \frac{\partial g_{22}}{\partial x^2} = \frac{\partial(4(x^2 + y^2))}{\partial y} = 8y. \end{aligned}$$

Thus, the only possibly non-zero Christoffel symbols (of both first and second kind) are the ones with indices

- 1,1,1, or

- 1,1,2, or
- 1,2,2, or
- 2,2,2

in some order.

Thus, the Christoffel symbols of first kind are given by

$$\Gamma_{111} = \frac{1}{2} \left(\frac{\partial g_{11}}{\partial x^1} + \frac{\partial g_{11}}{\partial x^1} - \frac{\partial g_{11}}{\partial x^1} \right) = \frac{1}{2} \frac{\partial g_{11}}{\partial x^1} = \frac{1}{2} \cdot 8x = 4x,$$

$$\Gamma_{112} = \Gamma_{121} = \frac{1}{2} \left(\frac{\partial g_{11}}{\partial x^2} + \frac{\partial g_{12}}{\partial x^1} - \frac{\partial g_{12}}{\partial x^1} \right) = \frac{1}{2} \frac{\partial g_{11}}{\partial x^2} = \frac{1}{2} \cdot 8y = 4y,$$

$$\Gamma_{211} = -\frac{1}{2} \frac{\partial g_{11}}{\partial x^2} = -\frac{1}{2} \cdot 8y = -4y$$

$$\Gamma_{212} = \Gamma_{221} = \frac{1}{2} \frac{\partial g_{22}}{\partial x^1} = \frac{1}{2} \cdot 8x = 4x,$$

$$\Gamma_{122} = -\frac{1}{2} \frac{\partial g_{22}}{\partial x^1} = -\frac{1}{2} \cdot 8x = -4x$$

$$\Gamma_{222} = \frac{1}{2} \frac{\partial g_{22}}{\partial x^2} = \frac{1}{2} \cdot 8y = 4y,$$

$$\Gamma_{ikj} = 0, \quad \text{for all other values of } i, j, k.$$

(4) To find the Christoffel symbols of second kind Γ_{jk}^i we use the formula

$$\Gamma_{jk}^i = g^{i\ell} \Gamma_{\ell kj} = g^{i1} \Gamma_{1kj} + g^{i2} \Gamma_{2kj} + g^{i3} \Gamma_{3kj}.$$

Since

$$[g^{ij}] = \begin{pmatrix} \frac{1}{4(x^2+y^2)} & 0 & 0 \\ 0 & \frac{1}{4(x^2+y^2)} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

$$\Gamma_{11}^1 = g^{11} \Gamma_{111} = \frac{4x}{4(x^2+y^2)}, \quad \Gamma_{12}^1 = \Gamma_{21}^1 = g^{11} \Gamma_{112} = \frac{4y}{4(x^2+y^2)},$$

$$\Gamma_{11}^2 = g^{22} \Gamma_{211} = -\frac{4y}{4(x^2+y^2)}, \quad \Gamma_{12}^2 = \Gamma_{21}^2 = g^{22} \Gamma_{212} = \frac{4x}{4(x^2+y^2)},$$

$$\Gamma_{22}^1 = g^{11} \Gamma_{122} = -\frac{4x}{4(x^2+y^2)}, \quad \Gamma_{22}^2 = g^{22} \Gamma_{222} = \frac{4y}{4(x^2+y^2)},$$

$$\Gamma_{kj}^i = 0, \quad \text{for all other values of } i, j, k.$$

8.5 Given that the transformation law for the Christoffel symbol of the second kind is

$$\Gamma^i{}_{jk}{}' = L_\ell^{i'} L_{j'}^m L_{k'}^n \Gamma_{mn}^\ell + L_m^{i'} L_{k'}^n \frac{\partial L_{j'}^m}{\partial x^n},$$

show that the covariant derivative of a covariant vector is a second-rank covariant tensor.

[Hint: You can use the equalities $L_{k'}^m = \frac{\partial x^m}{\partial x'^k}$ and $L_{j'}^r L_\ell^{j'} = \delta_{r,\ell}$ that we already have shown.]

Solution: We have

$$\begin{aligned} A_{i,k}{}' &= \frac{\partial A_i'}{\partial x'^k} - \Gamma_{ik}{}^j{}' A_j' \\ &= \frac{\partial}{\partial x^m} (L_{i'}^\ell A_\ell) \frac{\partial x^m}{\partial x'^k} - L_{j'}^r A_r \left(L_\ell^{j'} L_{i'}^m L_{k'}^n \Gamma_{mn}^\ell + L_m^{j'} L_{k'}^n \frac{\partial L_{i'}^m}{\partial x^n} \right) \\ &= L_{k'}^m \frac{\partial}{\partial x^m} (L_{i'}^\ell A_\ell) - (L_{j'}^r L_\ell^{j'}) L_{i'}^m L_{k'}^n \Gamma_{mn}^\ell A_r - (L_{j'}^r L_m^{j'}) L_{k'}^n A_r \frac{\partial L_{i'}^m}{\partial x^n} && \left(\text{as } L_{k'}^m = \frac{\partial x^m}{\partial x'^k} \right) \\ &= L_{k'}^m \frac{\partial}{\partial x^m} (L_{i'}^\ell A_\ell) - (\delta_{r,\ell}^r) L_{i'}^m L_{k'}^n \Gamma_{mn}^\ell A_r - (\delta_{r,m}^r) L_{k'}^n A_r \frac{\partial L_{i'}^m}{\partial x^n} && \left(\text{as } L_{j'}^r L_\ell^{j'} = \delta_{r,\ell} \right) \\ &= L_{k'}^m \frac{\partial}{\partial x^m} (L_{i'}^\ell A_\ell) - L_{i'}^m L_{k'}^n \Gamma_{mn}^\ell A_\ell - L_{k'}^n A_m \frac{\partial L_{i'}^m}{\partial x^n} \\ &= L_{k'}^m L_{i'}^\ell \frac{\partial A_\ell}{\partial x^m} + L_{k'}^m A_\ell \frac{\partial L_{i'}^\ell}{\partial x^m} - L_{i'}^m L_{k'}^n \Gamma_{mn}^\ell A_\ell - L_{k'}^n A_m \frac{\partial L_{i'}^m}{\partial x^n} \\ &= L_{k'}^m L_{i'}^\ell \frac{\partial A_\ell}{\partial x^m} + L_{k'}^n A_m \frac{\partial L_{i'}^m}{\partial x^n} - L_{i'}^m L_{k'}^n \Gamma_{mn}^\ell A_\ell - L_{k'}^n A_m \frac{\partial L_{i'}^m}{\partial x^n} && \left(\text{as } m, \ell \text{ are dummy suffices} \right) \\ &= L_{k'}^m L_{i'}^\ell \frac{\partial A_\ell}{\partial x^m} - L_{i'}^m L_{k'}^n \Gamma_{mn}^\ell A_\ell \\ &= L_{i'}^\ell L_{k'}^m \frac{\partial A_\ell}{\partial x^m} - L_{i'}^\ell L_{k'}^m \Gamma_{\ell m}^n A_n && \left(\text{swapping dummy suffices} \right) \\ &= L_{i'}^\ell L_{k'}^m \left(\frac{\partial A_\ell}{\partial x^m} - \Gamma_{\ell m}^n A_n \right) \\ &= L_{i'}^\ell L_{k'}^m A_{\ell,m}. \end{aligned}$$