Tensor Analysis – Practical 8

Information:

- Please make sure to complete **all** exercises **before** the next lecture.
- The exercises marked with [**See lecture**] were solved in class.
- The exercises are **not organised by difficulty**.

8.1 The Christoffel symbols of the first kind can be expressed in terms of the metric tensor as follows:

$$\Gamma_{ijk} = \frac{1}{2}\left(\frac{\partial g_{ik}}{\partial x^j} + \frac{\partial g_{ij}}{\partial x^k} - \frac{\partial g_{kj}}{\partial x^i}\right).$$

This expression simplifies for orthogonal coordinate systems (those with $g_{ij} = 0$ when $i \neq j$). Find the simplified expression for Γ_{ijk} in the cases

(1) $i = j = k$;

(2) $i = j \neq k$;

(3) $i \neq j = k$;

(4) i, j, k are all distinct.

8.2 [**See lecture**] We have seen in Practical 5 spherical coordinates $(x^1, x^2, x^3) = (r, \phi, \theta)$, the metric tensors are

$$g_{11} = 1, \quad g_{22} = r^2, \quad g_{33} = r^2 \sin^2 \phi,$$

and the other components of the metric tensor are trivial. Find the Christoffel symbols of the first and second kind.

8.3 Consider the **2D** coordinates $(x^1, x^2) = (\rho, \theta)$, in which the position vector $\mathbf{r}$ is given by

$$\mathbf{r} = \rho^2 \cos(2\theta)\, \mathbf{i}_1 + \rho^2 \sin(2\theta)\, \mathbf{i}_2,$$

where $\mathbf{i}_1$, $\mathbf{i}_2$ are the usual 2-dimensional Cartesian basis vectors.

(1) Find the corresponding basis vectors $\mathbf{e}_1$ and $\mathbf{e}_2$.

(2) Find the covariant metric tensor g_{ij} and give your answer as a 2×2 matrix. Using the fact that the basis $\mathbf{e}_1, \mathbf{e}_2$ is orthogonal, find the contravariant metric tensor g^{ij}.

(3) Find all Christoffel symbols Γ_{ijk} of the first kind.

(4) Find all Christoffel symbols Γ^i_{jk} of the second kind.

8.4 Consider the coordinates $(x^1, x^2, x^3) = (x, y, t)$, in which the position vector $\mathbf{r}$ is given by

$$\mathbf{r} = (x^2 - y^2)\,\mathbf{i}_1 + (2xy)\,\mathbf{i}_2 + t\,\mathbf{i}_3,$$

where $\mathbf{i}_1$, $\mathbf{i}_2$, $\mathbf{i}_3$ are the usual Cartesian basis vectors.

(*1*) Find the corresponding basis vectors $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$.

(*2*) Find the covariant metric tensor g_{ij} and the contravariant metric tensor g^{ij}. Give your final answer as 2×2 matrices.

(*3*) Find all Christoffel symbols Γ_{ijk} of the first kind.

(*4*) Find all Christoffel symbols Γ^i_{jk} of the second kind.

8.5 Given that the transformation law for the Christoffel symbol of the second kind is

$$\Gamma^i{}_{jk}{}' = L^{i'}_{\ell} L^m_{j'} L^n_{k'} \Gamma^{\ell}{}_{mn} + L^{i'}_m L^n_{k'} \frac{\partial L^m_{j'}}{\partial x^n},$$

show that the covariant derivative of a covariant vector is a second-rank covariant tensor.

[Hint: You can use the equalities $L^m_{k'} = \frac{\partial x^m}{\partial x'^k}$ and $L^r_{j'} L^{j'}_{\ell} = \delta^r_{\cdot\ell}$ that we already have shown.]