Tensor Analysis – Practical 9

Solutions ______________________________

Information:

- Please make sure to complete **all** exercises **before** the next lecture.
- The exercises marked with [**See lecture**] were solved in class.
- The exercises are **not organised by difficulty**.

9.1 Given a 2-dimensional metric tensor

$$[g_{ij}] = \begin{pmatrix} 1 & 0 \\ 0 & e^{2x} \end{pmatrix}$$

for the a coordinate system $(x^1, x^2) = (x, y)$, calculate the components of the Ricci tensor R_{ij}.

Solution:

Let us find the Christoffel symbols of second kind. The only non-zero partial derivative $\frac{\partial g_{ik}}{\partial x^\ell}$ is

$$\frac{\partial g_{22}}{\partial x^1} = \frac{\partial e^{2x}}{\partial x} = 2e^{2x}.$$

Thus, using the formula

$$\Gamma^i{}_{jk} = \frac{1}{2} g^{i\ell} \left(\frac{\partial g_{\ell j}}{\partial x^k} + \frac{\partial g_{\ell k}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^\ell} \right),$$

we see that the only possibly non-zero Christoffel symbols of second type are the ones having two indices 2 and one index 1. That is, Γ^1_{22}, Γ^2_{12}, and Γ^2_{21}.

Before computing the Christoffel symbols, we need to compute the inverse metric g^{ij}:

$$[g^{ij}] = \begin{pmatrix} 1 & 0 \\ 0 & e^{-2x} \end{pmatrix}$$

Thus,

$$\Gamma^1_{11} = \Gamma^1_{12} = \Gamma^1_{21} = \Gamma^2_{11} = \Gamma^2_{22} = 0$$

and

$$\begin{aligned}
\Gamma^1_{22} &= \frac{1}{2}g^{11}\left(\frac{\partial g_{12}}{\partial x^2} + \frac{\partial g_{12}}{\partial x^2} - \frac{\partial g_{22}}{\partial x^1}\right) \\
&= -\frac{1}{2}g^{11}\left(\frac{\partial g_{22}}{\partial x^1}\right) \\
&= -\frac{1}{2}\left(2e^{2x}\right) \\
&= -e^{2x}, \\
\Gamma^2_{12} &= \Gamma^2_{21} = \frac{1}{2}g^{22}\left(\frac{\partial g_{21}}{\partial x^2} + \frac{\partial g_{22}}{\partial x^1} - \frac{\partial g_{12}}{\partial x^2}\right) \\
&= \frac{1}{2}g^{22}\left(\frac{\partial g_{22}}{\partial x^1}\right) \\
&= \frac{1}{2}e^{-2x}\left(2e^{2x}\right) \\
&= 1.
\end{aligned}$$

Using the formula

$$R_{ij} = \frac{\partial \Gamma^k_{ij}}{\partial x^k} - \frac{\partial \Gamma^k_{ik}}{\partial x^j} + \Gamma^k_{ij}\Gamma^\ell_{k\ell} - \Gamma^\ell_{ik}\Gamma^k_{j\ell}$$

we can compute the Ricci tensor components

$$
\begin{aligned}
R_{11} &= \frac{\partial\Gamma^k_{11}}{\partial x^k} - \frac{\partial\Gamma^k_{1k}}{\partial x^1} + \Gamma^k_{11}\Gamma^\ell_{k\ell} - \Gamma^\ell_{1k}\Gamma^k_{1\ell} \\
&= -\frac{\partial\Gamma^k_{1k}}{\partial x^1} - \Gamma^\ell_{1k}\Gamma^k_{1\ell} \qquad\qquad\qquad (\Gamma^k_{11} = 0) \\
&= -\frac{\partial\Gamma^1_{11}}{\partial x^1} + \frac{\partial\Gamma^2_{12}}{\partial x^1} - \Gamma^\ell_{11}\Gamma^1_{1\ell} - \Gamma^\ell_{12}\Gamma^2_{1\ell} \\
&= \frac{\partial\Gamma^2_{12}}{\partial x^1} - \Gamma^\ell_{12}\Gamma^2_{1\ell} \\
&= \frac{\partial\Gamma^2_{12}}{\partial x^1} - \Gamma^1_{12}\Gamma^2_{11} - \Gamma^2_{12}\Gamma^2_{12} \\
&= \frac{\partial\Gamma^2_{12}}{\partial x^1} - \Gamma^2_{12}\Gamma^2_{12} \\
&= \frac{\partial 1}{\partial x} - 1\cdot 1 = -1, \\
R_{12}R_{21} &= \frac{\partial\Gamma^k_{12}}{\partial x^k} - \frac{\partial\Gamma^k_{1k}}{\partial x^2} + \Gamma^k_{12}\Gamma^\ell_{k\ell} - \Gamma^\ell_{1k}\Gamma^k_{2\ell} \\
&= \frac{\partial\Gamma^k_{12}}{\partial x^k} + \Gamma^k_{12}\Gamma^\ell_{k\ell} - \Gamma^\ell_{1k}\Gamma^k_{2\ell} \qquad\qquad\qquad \left(\frac{\partial\Gamma^m_{np}}{\partial x^2} = 0\right), \\
&= \frac{\partial\Gamma^1_{12}}{\partial x^1} + \frac{\partial\Gamma^2_{12}}{\partial x^2} + \Gamma^k_{12}\Gamma^\ell_{k\ell} - \Gamma^\ell_{1k}\Gamma^k_{2\ell} \\
&= \Gamma^k_{12}\Gamma^\ell_{k\ell} - \Gamma^\ell_{1k}\Gamma^k_{2\ell} \\
&= \Gamma^1_{12}\Gamma^\ell_{1\ell} + \Gamma^2_{12}\Gamma^\ell_{2\ell} - \Gamma^\ell_{11}\Gamma^1_{2\ell} - \Gamma^\ell_{12}\Gamma^2_{2\ell} \\
&= \Gamma^2_{12}\Gamma^\ell_{2\ell} - \Gamma^\ell_{12}\Gamma^2_{2\ell} \\
&= \Gamma^2_{12}\Gamma^1_{21} + \Gamma^2_{12}\Gamma^2_{22} - \Gamma^1_{12}\Gamma^2_{21} - \Gamma^2_{12}\Gamma^2_{22} = 0, \\
R_{22} &= \frac{\partial\Gamma^k_{22}}{\partial x^k} - \frac{\partial\Gamma^k_{2k}}{\partial x^2} + \Gamma^k_{22}\Gamma^\ell_{k\ell} - \Gamma^\ell_{2k}\Gamma^k_{2\ell} \\
&= \frac{\partial\Gamma^k_{22}}{\partial x^k} + \Gamma^k_{22}\Gamma^\ell_{k\ell} - \Gamma^\ell_{2k}\Gamma^k_{2\ell} \\
&= \frac{\partial\Gamma^1_{22}}{\partial x^1} + \frac{\partial\Gamma^2_{22}}{\partial x^2} + \Gamma^k_{22}\Gamma^\ell_{k\ell} - \Gamma^\ell_{2k}\Gamma^k_{2\ell} \\
&= \frac{\partial(-e^{2x})}{\partial x} + \Gamma^k_{22}\Gamma^\ell_{k\ell} - \Gamma^\ell_{2k}\Gamma^k_{2\ell} \\
&= -2e^{2x} + \Gamma^k_{22}\Gamma^1_{k1} + \underline{\Gamma^k_{22}\Gamma^2_{k2}} - \Gamma^1_{2k}\Gamma^k_{21} - \underline{\Gamma^2_{2k}\Gamma^k_{22}} \\
&= -2e^{2x} + \Gamma^k_{22}\Gamma^2_{k2} \\
&= -2e^{2x} + \Gamma^1_{22}\Gamma^2_{12} + \Gamma^2_{22}\Gamma^2_{22} \\
&= -2e^{2x} + \Gamma^1_{22}\Gamma^2_{12} = -2e^{2x} - (-e^{-2x})\cdot 1 = -e^{-2x}.
\end{aligned}
$$

9.2 Consider the 3-dimensional space in spherical coordinates that has arc length element

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2$$

Compute the Ricci tensor components R_{kk} for this coordinate system. Here $(x^1, x^2, x^3) = (r, \theta, \phi)$

Solution: We have

$$[g_{ij}] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{pmatrix}$$

Thus, the only non-zero partial derivative $\frac{\partial g_{ik}}{\partial x^\ell}$ are

$$\frac{\partial g_{22}}{\partial x^1} = \frac{\partial r^2}{\partial r} = 2r$$
$$\frac{\partial g_{33}}{\partial x^1} = \frac{\partial r^2 \sin^2\theta}{\partial r} = 2r\sin^2\theta$$
$$\frac{\partial g_{33}}{\partial x^2} = \frac{\partial r^2 \sin^2\theta}{\partial \theta} = 2r^2 \sin\theta\cos\theta$$

Thus, the only possible non-zero Christoffel symbols are the ones having two indices 2 and one index 1, the ones having two indices 3 and one index 1, and the ones having two indices 3 and one index 2.

Before finding them, we still need

$$[g^{ij}] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & \frac{1}{r^2\sin^2\theta} \end{pmatrix}.$$

Thus, using the formula

$$\Gamma^i_{\ jk} = \frac{1}{2} g^{i\ell}\left(\frac{\partial g_{\ell j}}{\partial x^k} + \frac{\partial g_{\ell k}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^\ell}\right),$$

we get that the non-zero Christoffel symbols are

$$\Gamma^1_{22} = r, \quad \Gamma^1_{33} = -r\sin^2\theta$$
$$\Gamma^2_{12}\Gamma^2_{21} = -\frac{1}{r}, \quad \Gamma^2_{33} = -\sin\theta\cos\theta$$
$$\Gamma^3_{13} = \frac{1}{r}, \quad \Gamma^3_{23} = \cot\theta.$$

Using the formula

$$R_{ij} = \frac{\partial \Gamma^k_{ij}}{\partial x^k} - \frac{\partial \Gamma^k_{ik}}{\partial x^j} + \Gamma^k_{ij}\Gamma^l_{k\ell} - \Gamma^\ell_{ik}\Gamma^k_{j\ell}$$

we can show that the Ricci tensor components are

$$R_{11} = 0, \quad R_{22} = 1, \quad R_{33} = \sin^2\theta.$$

9.3 Given a 3-dimensional metric tensor

$$g_{ij} = \begin{pmatrix} e^{2u} & 0 & 0 \\ 0 & e^{2v} & 0 \\ 0 & 0 & e^{2w} \end{pmatrix}$$

in coordinates $(x^1, x^2, x^3) = (u, v, w)$. Find the Ricci tensor components R_{ij}.

Solution: Again,

$$g_{ij} = \begin{pmatrix} e^{2u} & 0 & 0 \\ 0 & e^{2v} & 0 \\ 0 & 0 & e^{2w} \end{pmatrix}.$$

Thus, the only non-zero partial derivative $\frac{\partial g_{ik}}{\partial x^\ell}$ are

$$\frac{\partial g_{11}}{\partial x^1} = \frac{\partial e^{2u}}{\partial u} = 2e^{2u},$$
$$\frac{\partial g_{22}}{\partial x^2} = \frac{\partial e^{2v}}{\partial v} = 2e^{2v},$$
$$\frac{\partial g_{33}}{\partial x^3} = \frac{\partial e^{2w}}{\partial w} = 2e^{2w}.$$

Thus, the only possible non-zero Christoffel symbols are the ones having three repeated indices.

Before computing these Christoffel symbols, we still need

$$g^{ij} = \begin{pmatrix} e^{-2u} & 0 & 0 \\ 0 & e^{-2v} & 0 \\ 0 & 0 & e^{-2w} \end{pmatrix}.$$

Thus, using the formula

$$\Gamma^i{}_{jk} = \frac{1}{2} g^{i\ell} \left(\frac{\partial g_{\ell j}}{\partial x^k} + \frac{\partial g_{\ell k}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^\ell} \right),$$

we get that the non-zero Christoffel symbols are

$$\Gamma^1_{11} = \frac{1}{2}(e^{-2u})(2e^{2u}) = 1,$$
$$\Gamma^2_{22} = \frac{1}{2}(e^{-2v})(2e^{2v}) = 1,$$
$$\Gamma^3_{33} = \frac{1}{2}(e^{-2w})(2e^{2w}) = 1.$$

Therefore, if we apply this to the formula

$$R_{ij} = \frac{\partial \Gamma^k_{ij}}{\partial x^k} - \frac{\partial \Gamma^k_{ik}}{\partial x^j} + \Gamma^k_{ij}\Gamma^l_{k\ell} - \Gamma^\ell_{ik}\Gamma^k_{j\ell}$$

we obtain

$$R_{ij} = 0, \text{ for all } i, j = 1, 2.$$

Revision exercises

9.4 Consider the coordinate system having coordinates $(x^1, x^2, x^3) = (\phi, \theta, z)$ and position vector

$$\mathbf{r} = \cos\phi(\sin\theta + \cos\theta)\mathbf{i}_1 + \sin\phi(\sin\theta + \cos\theta)\mathbf{i}_2 + z\mathbf{i}_3,$$

where $\mathbf{i}_1$, $\mathbf{i}_2$, $\mathbf{i}_3$ are the usual Cartesian basis vectors.

(*1*) Find basis vectors $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_2$.

(*2*) Show that this is an orthogonal coordinate system.

Solution: **(1)** To find the basis vectors $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ we use the formula

$$\mathbf{e}_i = \frac{\partial \mathbf{r}}{\partial x^i}.$$

We have

$$\begin{aligned}\mathbf{e}_1 &= \frac{\partial \mathbf{r}}{\partial x^1} = \frac{\partial}{\partial \phi}(\cos\phi(\sin\theta + \cos\theta)\mathbf{i}_1 + \sin\phi(\sin\theta + \cos\theta)\mathbf{i}_2 + z\mathbf{i}_3)\\ &= -\sin\phi(\sin\theta + \cos\theta)\mathbf{i}_1 + \cos\phi(\sin\theta + \cos\theta)\mathbf{i}_2,\end{aligned}$$

$$\begin{aligned}\mathbf{e}_2 &= \frac{\partial \mathbf{r}}{\partial x^2} = \frac{\partial}{\partial \theta}(\cos\phi(\sin\theta + \cos\theta)\mathbf{i}_1 + \sin\phi(\sin\theta + \cos\theta)\mathbf{i}_2 + z\mathbf{i}_3)\\ &= \cos\phi(\cos\theta - \sin\theta)\mathbf{i}_1 + \sin\phi(\cos\theta - \sin\theta)\mathbf{i}_2,\end{aligned}$$

$$\begin{aligned}\mathbf{e}_3 &= \frac{\partial \mathbf{r}}{\partial x^3} = \frac{\partial}{\partial \phi}(\cos\phi(\sin\theta + \cos\theta)\mathbf{i}_1 + \sin\phi(\sin\theta + \cos\theta)\mathbf{i}_2 + z\mathbf{i}_3)\\ &= \mathbf{i}_3.\end{aligned}$$

(2) To show that this is a orthogonal coordinate system, we need to show $\mathbf{e}_i \cdot \mathbf{e}_j = 0$, whenever $i \neq j$. In fact

$$\begin{aligned}\mathbf{e}_1 \cdot \mathbf{e}_2 &= \begin{pmatrix} -\sin\phi(\sin\theta + \cos\theta) \\ \cos\phi(\sin\theta + \cos\theta) \\ 0 \end{pmatrix} \cdot \begin{pmatrix} \cos\phi(\cos\theta - \sin\theta) \\ \sin\phi(\cos\theta - \sin\theta) \\ 0 \end{pmatrix} \\ &= -\sin\phi\cos\phi(\sin\theta + \cos\theta)(\cos\theta - \sin\theta) + \sin\phi\cos\phi(\sin\theta + \cos\theta)(\cos\theta - \sin\theta) \\ &= 0,\end{aligned}$$

$$\mathbf{e}_1 \cdot \mathbf{e}_3 = \begin{pmatrix} -\sin\phi(\sin\theta + \cos\theta) \\ \cos\phi(\sin\theta + \cos\theta) \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 0,$$

$$\mathbf{e}_2 \cdot \mathbf{e}_3 = \begin{pmatrix} \cos\phi(\cos\theta - \sin\theta) \\ \sin\phi(\cos\theta - \sin\theta) \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 0.$$

9.5 Again, consider the coordinate system of Question 9.4.

Find the covariant components of the vector

$$\mathbf{A} = -\mathbf{i}_1 + \mathbf{i}_2$$

with respect to the basis $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ of Question 9.4.

Solution: The covariant components of $\mathbf{A}$ are given by $A_i = \mathbf{A} \cdot \mathbf{e}_i$. Thus,

$$\begin{aligned}A_1 &= \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} -\sin\phi(\sin\theta + \cos\theta) \\ \cos\phi(\sin\theta + \cos\theta) \\ 0 \end{pmatrix} \\ &= \sin\phi(\sin\theta + \cos\theta) + \cos\phi(\sin\theta + \cos\theta) \\ &= (\sin\phi + \cos\phi)(\sin\theta + \cos\theta), \\ A_2 &= \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} \cos\phi(\cos\theta - \sin\theta) \\ \sin\phi(\cos\theta - \sin\theta) \\ 0 \end{pmatrix} \\ &= -\cos\phi(\cos\theta - \sin\theta) + \sin\phi(\cos\theta - \sin\theta) \\ &= (\sin\phi - \cos\phi)(\cos\theta - \sin\theta), \\ A_3 &= \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 0.\end{aligned}$$

9.6 Compute the metric coefficients of the arc length and the components of the covariant metric tensor g_{ii} for the coordinate system of Question 9.4.

Solution: Recall that The element of the arc length of an orthogonal system is

$$(ds)^2 = g_{ij}dx^i dx^j = g_{11}dx^1dx^1 + g_{22}dx^2dx^2 + g_{33}dx^3dx^3.$$

Let us then compute the components of the covariant metric tensor g_{ii}.

By definition $g_{ii} = \mathbf{e}_i \cdot \mathbf{e}_i$. Thus,

$$\begin{aligned} g_{11} = \mathbf{e}_1 \cdot \mathbf{e}_1 &= \begin{pmatrix} -\sin\phi(\sin\theta + \cos\theta) \\ \cos\phi(\sin\theta + \cos\theta) \\ 0 \end{pmatrix} \cdot \begin{pmatrix} -\sin\phi(\sin\theta + \cos\theta) \\ \cos\phi(\sin\theta + \cos\theta) \\ 0 \end{pmatrix} \\ &= \sin^2\phi(\sin\theta + \cos\theta)^2 + \cos^2\phi(\sin\theta + \cos\theta)^2 \\ &= (\sin^2\phi + \cos^2\phi)(\sin\theta + \cos\theta)^2 \\ &= (\sin\theta + \cos\theta)^2 \end{aligned}$$

$$\begin{aligned} g_{22} = \mathbf{e}_2 \cdot \mathbf{e}_2 &= \begin{pmatrix} \cos\phi(\cos\theta - \sin\theta) \\ \sin\phi(\cos\theta - \sin\theta) \\ 0 \end{pmatrix} \cdot \begin{pmatrix} \cos\phi(\cos\theta - \sin\theta) \\ \sin\phi(\cos\theta - \sin\theta) \\ 0 \end{pmatrix} \\ &= \cos^2\phi(\cos\theta - \sin\theta)^2 + \sin^2\phi(\cos\theta - \sin\theta)^2 \\ &= (\cos^2\phi + \sin^2\phi)(\cos\theta - \sin\theta)^2 \\ &= (\cos\theta - \sin\theta)^2 \end{aligned}$$

$$g_{33} = \mathbf{e}_3 \cdot \mathbf{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 1.$$

We are now ready to compute the metric coefficients of the arc length:

$$\begin{aligned} (ds)^2 &= g_{11}dx^1dx^1 + g_{22}dx^2dx^2 + g_{33}dx^3dx^3 \\ &= (\sin\theta + \cos\theta)^2dx^1dx^1 + (\cos\theta - \sin\theta)^2x^2dx^2 + dx^3dx^3 \\ &= ((\sin\theta + \cos\theta)dx^1)^2 + ((\cos\theta - \sin\theta)dx^2)^2 + (dx^3)^2. \end{aligned}$$

Now, since we also have

$$(ds)^2 = (h_1dx^1)^2 + (h_2dx^2)^2 + (h_3dx^3)^2,$$

the metric coefficients are

$$h_1 = \sin\theta + \cos\theta, \qquad h_2 = \cos\theta - \sin\theta, \qquad h_3 = 1.$$

9.7 Determine all Christoffel symbols of the first and of the second kind for the coordinate system of Question 9.4.

Solution: In Question 9.6, we computed

$$g_{11} = (\sin\theta + \cos\theta)^2, \qquad g_{22} = (\cos\theta - \sin\theta)^2, \qquad g_{33} = 1,$$

hence

$$g^{11} = \frac{1}{(\sin\theta + \cos\theta)^2}, \qquad g^{22} = \frac{1}{(\cos\theta - \sin\theta)^2}, \qquad g^{33} = 1.$$

Thus, the only non-zero partial derivative $\frac{\partial g_{ik}}{\partial x^\ell}$ are

$$\frac{\partial g_{11}}{\partial x^2} = \frac{\partial}{\partial\theta}(\sin\theta + \cos\theta)^2 = 2(\sin\theta + \cos\theta)(\cos\theta - \sin\theta) = 2(\cos^2\theta - \sin^2\theta),$$
$$\frac{\partial g_{22}}{\partial x^2} = \frac{\partial}{\partial\theta}(\cos\theta - \sin\theta)^2 = 2(\cos\theta - \sin\theta)(-\sin\theta - \cos\theta) = -2(\cos^2\theta - \sin^2\theta)$$

Analysing the formula

$$\Gamma_{ijk} = \frac{1}{2}\left(\frac{\partial g_{ik}}{\partial x^j} + \frac{\partial g_{ij}}{\partial x^k} - \frac{\partial g_{kj}}{\partial x^i}\right)$$

we conclude that the only possibly non-zero Christoffel symbols of first kind are the ones having two indices 1 and one index 2 or three indices 2.

Let us compute the non-zero Christoffel symbols of first kind.

$$\Gamma_{211} = \frac{1}{2}\left(\frac{\partial g_{21}}{\partial x^1} + \frac{\partial g_{21}}{\partial x^1} - \frac{\partial g_{11}}{\partial x^2}\right) = -\frac{1}{2}\frac{\partial g_{11}}{\partial x^2}$$
$$= -\frac{1}{2}2(\cos^2\theta - \sin^2\theta) = \sin^2\theta - \cos^2\theta$$

$$\Gamma_{222} = \frac{1}{2}\left(\frac{\partial g_{22}}{\partial x^2} + \frac{\partial g_{22}}{\partial x^2} - \frac{\partial g_{22}}{\partial x^2}\right) = \frac{1}{2}\left(\frac{\partial g_{22}}{\partial x^2}\right)$$
$$= \frac{1}{2}(-2(\cos^2\theta - \sin^2\theta)) = \sin^2\theta - \cos^2\theta.$$

Using Ricci's Theorem, we get

$$\Gamma_{211} + \Gamma_{121} = \frac{\partial g_{21}}{\partial x^1} = 0,$$

Thus,

$$\Gamma_{121} = -\Gamma_{211} = \cos^2\theta - \sin^2\theta.$$

Using that the Christoffel symbols are symmetric in the last two indices, we get

$$\Gamma_{112} = \Gamma_{121} = \cos^2\theta - \sin^2\theta.$$

We conclude

$$\begin{aligned}
\Gamma_{211} &= \sin^2\theta - \cos^2\theta\\
\Gamma_{222} &= \sin^2\theta - \cos^2\theta\\
\Gamma_{121} &= \cos^2\theta - \sin^2\theta\\
\Gamma_{112} &= \cos^2\theta - \sin^2\theta\\
\Gamma_{ijk} &= 0, \quad \text{for all other values of } (i,j,k).
\end{aligned}$$

To find the Christoffel symbols of second kind, we use the formula

$$\Gamma^i_{\ jk} = g^{i\ell}\Gamma_{\ell jk}.$$

Thus,

$$\begin{aligned}
\Gamma^2_{\ 11} &= g^{2\ell}\Gamma_{\ell 11} = \frac{1}{(\cos\theta - \sin\theta)^2}(\sin^2\theta - \cos^2\theta)\\
&= \frac{1}{(\cos\theta - \sin\theta)^2}(\sin\theta - \cos\theta)(\sin\theta + \cos\theta)\\
&= -\frac{\sin\theta + \cos\theta}{\cos\theta - \sin\theta}\\
\Gamma^2_{\ 22} &= g^{2\ell}\Gamma_{\ell 22} = \frac{1}{(\cos\theta - \sin\theta)^2}(\sin^2\theta - \cos^2\theta)\\
&= -\frac{\sin\theta + \cos\theta}{\cos\theta - \sin\theta}\\
\Gamma^1_{\ 21} &= g^{1\ell}\Gamma_{\ell 21} = \frac{1}{(\sin\theta + \cos\theta)^2}(\cos^2\theta - \sin^2\theta)\\
&= \frac{\cos\theta - \sin\theta}{\sin\theta + \cos\theta}\\
\Gamma^1_{\ 21} &= \Gamma^2_{\ 12} = \frac{\cos\theta - \sin\theta}{\sin\theta + \cos\theta}\\
\Gamma^i_{\ jk} &= 0, \quad \text{for all other values of } (i,j,k).
\end{aligned}$$

9.8 Determine the following Riemann-Christoffel tensors for the coordinate system of Question 9.4.

$$R^1_{\ 122}, \quad \text{and} \quad R^r_{\ ijk}.$$

Solution: We have that

$$R^r{}_{ijk} = \frac{\partial}{\partial x^j}\Gamma^r{}_{ki} - \frac{\partial}{\partial x^k}\Gamma^r{}_{ij} + \Gamma^p{}_{ik}\Gamma^r{}_{pj} - \Gamma^p{}_{ij}\Gamma^r{}_{pk}$$

Thus,

$$\begin{aligned} R^1{}_{122} &= \frac{\partial}{\partial x^2}\Gamma^1{}_{21} - \frac{\partial}{\partial x^2}\Gamma^1{}_{12} + \Gamma^p{}_{12}\Gamma^1{}_{p2} - \Gamma^p{}_{12}\Gamma^1{}_{p2} \\ &= \Gamma^p{}_{12}\Gamma^1{}_{p2} - \Gamma^p{}_{12}\Gamma^1{}_{p2} \\ &= \Gamma^1{}_{12}\Gamma^1{}_{12} + \Gamma^2{}_{12}\Gamma^1{}_{22} - \Gamma^1{}_{12}\Gamma^1{}_{12} - \Gamma^2{}_{12}\Gamma^1{}_{22} \\ &= 0. \end{aligned}$$

Moreover,

$$\begin{aligned} R^1{}_{112} &= \frac{\partial}{\partial x^1}\Gamma^1{}_{21} - \frac{\partial}{\partial x^2}\Gamma^1{}_{11} + \Gamma^p{}_{12}\Gamma^1{}_{p1} - \Gamma^p{}_{11}\Gamma^1{}_{p2} \\ &= \frac{\partial}{\partial x^1}\left(\frac{\cos\theta - \sin\theta}{\sin\theta + \cos\theta}\right) + \Gamma^1{}_{12}\Gamma^1{}_{11} + +\Gamma^2{}_{12}\Gamma^1{}_{21} - \Gamma^1{}_{11}\Gamma^1{}_{12} - \Gamma^2{}_{11}\Gamma^1{}_{22} \\ &= \frac{\partial}{\partial x^1}\left(\frac{\cos\theta - \sin\theta}{\sin\theta + \cos\theta}\right) \\ &= \frac{(-\sin\theta - \cos\theta)(\sin\theta + \cos\theta) - (\cos\theta - \sin\theta)(\cos\theta - \sin\theta)}{(\sin\theta + \cos\theta)^2} \\ &= \frac{-\sin^2\theta - 2\sin\theta\cos\theta - \cos^2\theta - \cos^2\theta + 2\sin\theta\cos\theta - \sin^2\theta}{(\sin\theta + \cos\theta)^2} \\ &= \frac{-2(\sin^2\theta + \cos^2\theta)}{\sin^2\theta + 2\sin\theta\cos\theta + \cos^2\theta} \\ &= \frac{-2}{1 + 2\sin\theta\cos\theta} \end{aligned}$$
