

Tensor Analysis – Practical 9

Information:

- Please make sure to complete **all** exercises **before** the next lecture.
- The exercises marked with [See lecture] were solved in class.
- The exercises are **not organised by difficulty**.

9.1 [See lecture] Consider coordinates $(x^1, x^2, x^3) = (u, v, w)$ with position vector given by

$$\mathbf{r} = e^u \mathbf{i}_1 + e^v \mathbf{i}_2 + w \mathbf{i}_3.$$

Using

$$\Gamma_{ijk} = \frac{1}{2} \left(\frac{\partial g_{ij}}{\partial x^k} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^i} \right)$$

find all Christoffel symbols of the first and second kind.

9.2 Consider the coordinate system (u, v, w) in \mathbb{R}^3 defined by the position vector

$$\mathbf{r}(u, v, w) = (e^u, \sin v, \ln(1 + w)).$$

Compute all Christoffel symbols of the first and second kind.

9.3 Suppose a 3D coordinate system $(x^1, x^2, x^3) = (a, b, c)$ is orthogonal with metric coefficients

$$h_1 = e^{a+b}, \quad h_2 = \sqrt{1 + a^2}, \quad h_3 = ab.$$

Compute all Christoffel symbols of the first and second kind.

9.4 Suppose a 3D coordinate system $(x^1, x^2, x^3) = (a, b, c)$ is used, and let S be the surface (two-dimensional) unit sphere parametrised by

$$\mathbf{r} = (\sin a \cos b, \sin a \sin b, \cos a), \quad 0 < a < \pi, \quad 0 < b < 2\pi.$$

- (a) Compute all Christoffel symbols of the second kind.
- (b) Using the Riemann–Christoffel tensor, show that S has non-zero curvature.

9.5 Let $S \subset \mathbb{R}^3$ be a (two-dimensional) surface parametrised by

$$\mathbf{r}(x, y) = (f(x) \cos y, f(x) \sin y, g(x)),$$

where f and g are smooth functions satisfying

$$\left(\frac{\partial f}{\partial x} \right)^2 + \left(\frac{\partial g}{\partial x} \right)^2 = 1.$$

Compute all Christoffel symbols of the first and second kind associated with this parametrisation.

9.6 Using the Riemann–Christoffel tensor, determine whether the surface S of the previous question has curvature zero. Assume that $f(x) \neq 0$ on the domain of the parametrisation.